

Thence the moment of momentum

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Fifty years have passed since Truesdell's seminal paper on the origin and status of the balance for the moment of momentum was published in ZAMM. It is time to take stock: Important new developments in the theory of generalized continua with internal degrees of freedom and some fascinating fundamental applications need to be pointed out. Is there new evidence from classical papers regarding its independence from the balance of linear momentum? Can micropolar theory be used to "explain" electromagnetism? How is the conservation of the moment of momentum viewed in today's physics textbooks? In this paper an attempt is made to answer these and many more interesting questions.

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1 | THOUGHTS ON A TRUESDELL PAPER IN ZAMM

More than 50 years ago Clifford A. Truesdell wrote two essays entitled "Die Entwicklung des Drallsatzes"¹ [69] and, more provocatively and written in English language, "Whence the Law of Moment of Momentum?" (in [70], pp. 239–271). There he poses a seemingly innocent question: Is the balance for the moment of momentum independent of Newton's *Lex Secunda*, *i.e.*, the conservation law of linear momentum or not. Both articles were written during the renaissance of modern higher continuum theories of the Cosserat type, which emphasize the possibility of additional rotational degrees of freedom of a material point independently of its translational momentum. There, on the continuum scale, a balance equation for the total angular momentum is formulated. The total angular momentum consists of the conventional moment of translational momentum plus a non-classical term, the so-called spin field, which is characteristic of the rotational degrees of freedom of a material point. The situation is analogous to the case of the total energy of a continuum, which is decomposed into a part for the kinetic energy, which is visible in the movement of the continuous body, and another one for the internal energy, invisible on the continuum level, so to speak. However, in contrast to angular momentum nobody claims that the balance of kinetic energy suffices to describe the energy contents of continuous matter, nor that the balance of internal energy (the First Law of Thermodynamics) is a consequence of Newton's Second Law.

Truesdell's analysis of the status of the balance for the moment of momentum culminates in radical conclusions:

- Physicists suffer from the misconception that Newton's Second Law can be used to derive the balance of angular momentum and do not comprehend that both are independent laws of nature. They are also unwilling to realize the need for their independence.

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¹ "The development of the law of angular momentum."

- Physicists embrace Newton's verbally expressed notions on motion uncritically. They ignore the fact that there are two independently valid laws of motion for a body that were first formulated in clear mathematical form by Euler many years after Newton's death.
- Physicists are also unaware about the development of generalized continuum theories, in particular micropolar media, which consider internal rotational degrees of freedom independently of translational ones.

In order to substantiate his claims Truesdell uses the monographs on theoretical physics by Joos (in English translation, [34]) and by Sommerfeld [63], two textbooks well known to physicists from the times of their educational training, even today. He also refers to Newton's Principia and to Manuscript V of Newton's miscellaneous essays, [21], as evidence that Newton did not really think in terms of moment of forces on extended bodies and hence could not even rudimentary grasp the importance of a dynamic law for the angular momentum. Finally he presents the work of Euler (in particular [12]) where the two laws of motion are clearly stated in mathematical form.

50 years after his essays it is time to draw a resume and to investigate such issues as:

- What is the current status of generalized continuum theories and how is the independence seen, expressed, and used for problem solving now?
- Did Newton truly disregard the notion of angular momentum and did he not consider extended (rigid) bodies? Is there some new historical evidence to which Truesdell already alluded in the English version of his paper?
- Did Euler explicitly claim and emphasize, not only between the lines, that the balances of momentum and angular momentum are not related? Did he distinguish between moment of momentum and total angular momentum?
- What is the status of the law of moment of momentum in today's physics education?

In what follows we seek to give answers based on (new) historic evidence and modern developments.

2 | AN ATTEMPT TO REVIEW THE FACTS

In order to review and to judge the results presented in the recent and in the classical literature objectively a few initial remarks are in order. It was mentioned that Truesdell wrote his papers during the renaissance of generalized continuum theories, in particular those of the Cosserat and micropolar type. Since then many new discoveries have been made. Among other things this led to some new continuum fields relevant in this context. Therefore in what follows we will first present the complete set of balances relevant to polar media. We will then use this information to put it in context with the literature from the past. Next the current situation of physics education is analyzed. Finally, two curious applications of the extended form of micropolar equations will be given: Curved motion under the absence of forces and moments and two "derivations" of Maxwell's equations, one based on linear momentum and one based on angular momentum or, more precisely, on spin.

Upfront the following is worthwhile mentioning: The local (total) energy density of a continuum is always clearly divided into a "visible" part known as kinetic energy (density) and a "hidden" one, known as internal energy (density). Unfortunately in the case of the angular momentum a similar terminology is rarely that clearly cut. In what follows, we propose the following wording. The (total) angular momentum density of a continuum is additively composed of the "visible" (density for the) moment of momentum and the one hidden on the macro-scale known as the spin (density). We proceed to substantiate this classification in mathematical terms.

2.1 | Balances

The thermomechanics of polar media is based on the balance laws for translational inertia, rotational inertia (also known as microinertia), (translational) momentum, angular momentum, and energy.² Some essential information in index notation can be found in the books of Eringen [7–10]. For a more modern form in invariant notation the reader should consult

²The balance and inequality of entropy will not be discussed here, first, because of conciseness and, second, because the derivation and restriction of constitutive equations is beyond the scope of this paper.

[5] or [75] and [77]. However, all of these books do not contain information on how to treat polar materials undergoing structural change that affects their microinertia. Such extended balances are presented in [31], albeit for the case of a spatial description, which embraces the idea of open systems, non-material regions, as well as “destructible particles.” This is a method of description particularly suited and used in the fluid mechanics world (see the extensive discussion and arguments provided in [31] and [55]). The idea of destructible material particles is useful for certain physics applications, as we will explain later in a suitable position by means of some examples. However, in view of the large solid mechanics community and their thinking in terms of bijective reference placement mappings we will concentrate in this paper on material bodies and (non-destructible) material particles when writing the aforementioned balance equations.

Recall upfront the nomenclature for a general balance law in the current placement for an additive quantity Ψ in a regular material domain $V(t)$, which reads

$$\frac{d\Psi}{dt} = \frac{d}{dt} \int_{V(t)} \rho \psi dV = - \oint_{\partial V(t)} \mathbf{n} \cdot \Phi dA + \int_{V(t)} (\pi + \zeta) dV, \quad (1)$$

where ψ is a (specific) tensorial density of rank n , Φ is a non-convective tensorial flux of rank $n + 1$, and π and ζ are volumetric production and supply tensorial densities of rank n , respectively. Sometimes the distinction between two volumetric quantities is not made, however, in order to quote from [49], p. 49: “Supply is different from production because it may be controlled from the exterior of” $V(t)$. \mathbf{n} is the unit outward normal on $\partial V(t)$.

In the context of balance equations the notions “conservation law” or “conserved quantity” frequently appear. In a relative abstract way they are established through the requirement of a vanishing four-divergence (which is the four-vector combination of the partial time derivative combined with the nabla operator) of that quantity (see [51], p. 52). Slightly more profane we can rephrase that as “the time derivative of a physical quantity is equal to the divergence of a flux” (by converting the surface integral in Equation (1) into a volume integral using the Gauss-Ostrogradsky theorem) plus a volumetric supply term (which can be “controlled”) or, straightforwardly by saying, a physical quantity is conserved if there are no productions present in the balance equation (see [49], Section 3.2.1.2).

By means of various integral and localization theorems one obtains the balance law in regular points as follows

$$\rho \frac{d\psi}{dt} + \nabla \cdot \Phi = \pi + \zeta, \quad \text{with the material time derivative } \frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + \mathbf{v} \cdot (\nabla \otimes \psi). \quad (2)$$

Consider now a material domain $V(t)$ that is cut into two regular parts, V^+ and V^- , by a singular surface I . Then, the surface of the volumetric region is given by $\partial V(t) = \Gamma^+ \cup \Gamma^- \cup \partial I$, where Γ^\pm are the surfaces of the regular regions V^\pm without I , and ∂I is the periphery of I . Non-surface related quantities such as ψ are allowed to be discontinuous across I , i.e., the limit values ψ^\pm from both regular regions V^\pm at the surface I can be different. The difference is captured by the jump operator, $\llbracket \psi \rrbracket = \psi^+ - \psi^-$. Note that for the sake of brevity, the singular surface shall not possess any intrinsic properties. Furthermore, I is not necessarily material, i.e., it may move at a velocity \mathbf{w} , which, in general, is different to the material velocity \mathbf{v} . The balance law in Equation (1) is still valid for each regular subdomain, V^+ and V^- . By adding these two balances the total balance is obtained. However, for the localization process generalized integral theorems are required, in order to incorporate the singular surface, see, e.g., [49]. Then, by means of the so-called pillbox argument, which is a special localization, the balance in singular points or “jump relation” in singular points arises

$$\mathbf{e} \cdot \llbracket (\mathbf{v} - \mathbf{w}) \otimes \rho \psi + \Phi \rrbracket = \mathbf{0}, \quad (3)$$

where \mathbf{e} is a unit normal vector of I pointing from V^- to V^+ . Truesdell and Toupin refer to this result (for the special case $\mathbf{w} = \mathbf{0}$) on pp. 526 in [71] as Kottchine’s theorem. They explicitly say that the volume terms from the supply ζ (which they call source of ψ) plays no part. However, their global balance has no production term, $\pi = \mathbf{0}$. Without explicitly saying so, they restrict themselves to balances of conserved quantities or to conservation laws for short. If we allow volumetric productions to be present, the limit process of the pillbox argument can lead to non-vanishing contributions from the volume integrals over π . In fact these can even be singular. An example for a non-conserved quantity for which this

theorem has to be modified is the internal energy, see [1], pg. 633. Thus, for simplicity, we will restrict our jump conditions exclusively to fully conserved quantities, which by definition do not carry production terms. Note that the most prominent singular surface is the boundary of the material body itself, ∂V .

At this point it is illustrative to recall for what reason the term “material” volume was introduced: By definition, matter, more specifically the amount of mass associated with it, cannot leave nor enter such a region. The amount of matter, in particular mass, is simply conserved. Now obviously the scalar mass density ρ is discontinuous across ∂V . However, this is not in contradiction to (3), because ∂V is, of course, material, such that $\mathbf{w} = \mathbf{v}$ and $\mathbf{e} = \mathbf{n}$. Then, the jump relation reduces to $\mathbf{n} \cdot \llbracket \Phi \rrbracket = 0$.³ And in this case $\Phi \equiv 0$, because a non-convective mass flux cannot exist. In order to conclude and to repeat from where we started from: This equation expresses mathematically that no mass can leave the material volume through its material surface.

Quite often, for example in [49], it is customary to complement these general equations now simply by a table, where all the field quantities of interest are compiled. Such an approach is rewarding if one focuses only on the elementary balances, see [50]. However, in the present case such a table easily becomes unwieldy. Moreover, the type of matter we wish to consider allows the material points to have translational as well as rotational degrees of freedom. In other words we are facing a non-standard, generalized continuum in the sense of [41] and [42], which requires somewhat less known continuum fields for its description. Therefore, from a didactic point of view, it is advisable to go through each of the balances separately. And, what is more, before that we will introduce the non-standard fields of generalized continua and state our objectives.

We are interested in describing the motion of a three-dimensional material continuous body, $V(t)$. Its material points, also referred to as material particles, possess, in a coupled manner, translational as well as rotational degrees of freedom. This becomes more obvious if we consider the following expression for the specific⁴ kinetic energy density of such a particle,

$$\mathcal{E}_{\text{kin}} = \frac{1}{2} \mathbf{v} \cdot \mathbf{I} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{B} \cdot \boldsymbol{\omega} + \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{J} \cdot \boldsymbol{\omega}. \quad (4)$$

The first term is the translational part of the specific kinetic energy in terms of the **translational velocity**, \mathbf{v} . \mathbf{I} denotes the unit tensor. If we multiply it by the “translational inertia” of the material points, which manifests itself in the traditional continuum field of **mass density**, ρ , we obtain the translational kinetic energy of this point per unit volume. On first glance, this may seem like a unnecessarily complicated way of writing the specific kinetic energy term. However, note that it was, first, done on purpose in order to stress its similarity to the two other contributions in the total specific energy density, which show this bilinear tensorial form from the very beginning. Second, recall the concept of transversal and longitudinal mass used in particle physics, *i.e.*, a resistance to translational speed dependent on the direction of velocity. Hence, there might be a need for a more generalized concept. On the other hand, “rotational inertia” is incorporated in the third term, namely in the field of the symmetric **microinertia tensor**, \mathbf{J} . This tensor is an offspring of the inertia tensor known from rigid body dynamics (see Equation (35)₁), which is described mathematically in terms of the three main axes of a rigid body. Analogously the non-classical continuum field \mathbf{J} can be mathematically described by a triad of three orientational vectors, the so-called directors. Originally the term “micropolar materials” was limited to non-deformable directors ([10], p. 33). Hence \mathbf{J} was “rigid” in the sense that the material particle could undergo only rigid body rotations. This terminology was relaxed later and a deformation of the directors was allowed leading to so-called “micromorphic continua” (see [7], Chapter 7, [8], Chapter 17). In this article micropolar matter is *a priori* allowed to have a non-rigid microinertia. In addition, the material points may undergo microstructural changes in terms of this microinertia, as we shall explain shortly in the passage, where the balance for microinertia will be discussed. Moreover, we refer to the “rotational” velocity $\boldsymbol{\omega}$ in the third term of (4) as the **angular velocity**. The latter is also known as the microrotation vector (see [7], p. 26).

The expression in the middle of Equation (4) is a term coupling translational and rotational kinetic energy. Originally it is also motivated from rigid body dynamics, see (37). The connecting quantity is another non-standard continuum field, the **coupling tensor** \mathbf{B} . In rigid body dynamics, see (35)₂, it is introduced as a skew-symmetric tensor. For the case of a rigid body the coupling tensor vanishes if the center of mass is chosen as a reference position. However, if it is insisted that a three-term expression for the kinetic energy stays valid for a material point of a deformable generalized continuum,

³ Because ρ is a scalar we must put $\psi \rightarrow \psi \equiv 1$ in the general equation (3).

⁴ This means a physical quantity per unit mass and not per unit volume.

\mathcal{B} can be used as an additional degree of freedom. Then it is also no longer required that it is a skew-symmetric tensor as in the case of a rigid body. As we shall see in the example section such a more general coupling tensor field \mathcal{B} makes it possible to describe helix-type of motions for particles free of forces and moments, in particular free of forces of the Lorentz type. This was originally demonstrated by [77], Section 3.2 in Russian and in condensed version on p. 585 in [54]. We will get back to that in our Example 2.4.

Finally, in the context of ρ , \mathbf{J} , and \mathcal{B} , it should be mentioned that in [31] particular emphasis was paid to the fact that these are continuum fields characteristic of a Representative Volume Element (RVE). This setting was called by the authors “macro or continuum level.” Within the RVE we are on a “mesolevel.” The RVE consists of a very high number of “particles,” each of which we refer to as the “microlevel.” Averaging procedures, one could refer to it as “homogenization,” were applied to relate the properties of the particles at the microlevel to the continuum fields. This has also been done for material points, see [59]. However, the arguments provided in Section 3 of [31] could in principle also be adjusted to cover this case.

Once accepted, the expression (4) for the specific kinetic energy can be used to define two kinematic quantities namely the specific **linear momentum**, \mathbf{p} , and the **dynamic⁵ spin**, \mathbf{s} , as follows:

$$\mathbf{p} := \frac{\partial \mathcal{E}_{\text{kin}}}{\partial \mathbf{v}} = \mathbf{v} + \mathcal{B} \cdot \boldsymbol{\omega}, \quad \mathbf{s} := \frac{\partial \mathcal{E}_{\text{kin}}}{\partial \boldsymbol{\omega}} = \mathbf{v} \cdot \mathcal{B} + \mathbf{J} \cdot \boldsymbol{\omega}. \quad (5)$$

The notion of dynamic spin deserves some further explanation. In Sedov [60] we find a somewhat intuitive interpretation of this new quantity on p. 153 even though his process of homogenization from the micro- to the continuum level remains somewhat foggy: “Consider a system, consisting of a nucleus and, revolving about it, an electron, *i.e.*, an atom. The electron revolves in its orbit with a velocity of the order of the speed of light. Therefore, regardless of the small size of the atom, the system nucleus-electron possesses a significant intrinsic angular momentum. The angular momentum, arising from the revolution of the electron in an orbit, is known as orbital angular momentum.⁶ Moreover, the electron, and likewise the nucleus, have an intrinsic angular momentum, namely a spin, the origin of which cannot be explained by the introduction of corresponding mechanical motion. In general, all atoms have, generally speaking, intrinsic angular momentum \mathbf{k} .⁷ However, in many cases, the random motion of the atoms causes the sum of these angular momenta over all atoms to vanish. On the other hand, however, the motion of the elementary particles may be ordered, for instance, by applying a magnetic field. Then the sum of the internal momenta of all atoms will differ from zero. In this case, the sum $K' = \int_V \mathbf{k} \rho \, d\tau$ of the intrinsic angular momenta must appear in the expression for the angular momentum of a macroscopic particle of a continuous medium.”

We will explain in detail below that the description of motion of such generalized or micropolar materials is based on two independent balances, (a) the balance for the specific linear momentum, \mathbf{p} , and (b) for the **specific angular momentum**, $\mathbf{x} \times \mathbf{p} + \mathbf{s}$.

And now we proceed to introduce the balances one by one.

- **Balances of mass** As explained before the mass of a material volume $V(t)$ is conserved and the global mass balance must read:

$$\frac{d}{dt} \int_{V(t)} \rho \, dV = 0. \quad (6)$$

Hence the local balances follow according to Equations (2) and (3),

$$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0, \quad \mathbf{e} \cdot \llbracket \rho(\mathbf{v} - \mathbf{w}) \rrbracket = 0. \quad (7)$$

- **Balances of microinertia** In order to obtain the balance of microinertia in local and global form we start at the local level for micropolar material particle. In a co-rotating, particle-attached frame, the quantities of which are identifiable

⁵ We decided to add this adjective in order not to confuse it with the notion of spin in quantum mechanics, although there may well be a connection, which we shall not explore in this paper.

⁶ In the way of speech of this paper this should be called “moment of momentum” (lever arm \times translational momentum). In German it is adequately called “Bahndrehimpuls”=“orbital moment of momentum.”

⁷ Because Sedov refers to “all atoms” this might correspond to our symbol $\mathbf{x} \times \mathbf{p} + \mathbf{s}$, orbital moment of momentum plus spin.

by a prime, we start from the following local kinetic equation of corresponding microrotation, \mathbf{J}' :

$$\frac{d\mathbf{J}'}{dt} = \chi_J'. \quad (8)$$

The symmetric tensor field χ_J' is the local production of microinertia in the co-rotating frame. Now recall the Poisson relation (see [75], p. 70, or [32]),

$$\frac{d\mathbf{Q}}{dt} = \boldsymbol{\omega} \times \mathbf{Q}, \quad (9)$$

where \mathbf{Q} is a proper orthogonal tensor of rotation, such that

$$\mathbf{J} = \mathbf{Q} \cdot \mathbf{J}' \cdot \mathbf{Q}^\top, \quad \chi_J = \mathbf{Q} \cdot \chi_J' \cdot \mathbf{Q}^\top. \quad (10)$$

where \mathbf{Q} is a proper orthogonal tensor. Hence we conclude that in regular points:

$$\frac{d\mathbf{J}}{dt} = \boldsymbol{\omega} \times \mathbf{J} - \mathbf{J} \times \boldsymbol{\omega} + \chi_J \quad (11)$$

and the corresponding jump conditions at singular interfaces without production of microinertia:

$$\mathbf{e} \cdot \llbracket (\mathbf{v} - \mathbf{w}) \otimes \rho \mathbf{J} \rrbracket = \mathbf{0}, \quad (12)$$

which is the correct singular balance, if the production of microinertia is ignored, $\chi_J \equiv \mathbf{0}$.

So far the balance of microinertia has only been written in its local variants. If one wishes to obtain a global statement, Equation (10) is to be multiplied by ρ and afterwards integrated over the whole material volume. The global balance of microinertia is then given by:

$$\frac{d}{dt} \int_{V(t)} \rho \mathbf{J} dV = \int_{V(t)} \rho (\boldsymbol{\omega} \times \mathbf{J} - \mathbf{J} \times \boldsymbol{\omega} + \chi_J) dV. \quad (13)$$

The production term deserves some further comments. It was pointed out already that originally the microinertia just underwent rigid body rotations and, consequently, there was no production, $\chi_J = \mathbf{0}$, see [10] (p. 13), [7] (p. 41). The potential need for this additional feature during modeling of structural changes and transitions was emphasized relatively early, for example Eringen in [7], p. 46. More recently this idea was reexamined in [25], p. 69 albeit only for formal reasons regarding the general structure of balances. In [31] the idea of describing structural change was detailed for open systems in spatial description. This is particularly valuable if particles are continuously dividing or agglomerating, for example in a crusher or during chemical reactions. They may enter and leave from one representative volume element to the other. This was investigated for example in [53–55]. However, also material particles may undergo structural change. As a potential application Eringen in [7], p. 46 refers to a study of dense suspensions in viscous fluids. Other examples of such behavior are phase transitions of polar media, where the mass of the particle is conserved but its shape and volume may change. A particular example of this are phase transitions in liquid crystals. In this context the pioneering work of Leslie (see for example [66]) should be mentioned. However, this theory is not formulated in terms of a general microinertia. Rather a “digit” in terms of a single rigid director is used on the continuum level to describe the additional rotational degree of freedom. The digit can rotate but it cannot change its shape. The \mathbf{J} -concept could provide a straightforward generalization. Such an attempt to bridge the gap between both approaches was provided by Eringen in [6]. However, a detailed descriptive theory of nematic crystals using the microinertia \mathbf{J} and its production χ_J was not presented until today. Finally, as a last example of structural changes of material particles, consider the dielectric polarization and reorientation of the constituting sub-particles within the material element due to external electromagnetic fields (see [72]).

- **Balance of the coupling tensor** The coupling tensor \mathbf{B} obeys a balance similar to the one shown in (11):

$$\frac{d\mathbf{B}}{dt} + \mathbf{B} \times \boldsymbol{\omega} - \boldsymbol{\omega} \times \mathbf{B} = \chi_B, \quad (14)$$

where χ_B is its production. This equation was more or less formally introduced ([31], p. 1773). Concrete physics based applications for the production χ_B are still missing.

- **Balances of momentum** For a generalized micropolar continuum the balance of momentum reads in global form,

$$\frac{d}{dt} \int_{V(t)} \rho \mathbf{p} dV = \oint_{\partial V(t)} \mathbf{n} \cdot \boldsymbol{\sigma} dA + \int_{V(t)} \rho (\mathbf{f} + \chi_B \cdot \boldsymbol{\omega}) dV. \quad (15)$$

This is the generalization of Newton's LEX II from the Principia to generalized media, see [37] (in Latin) or [56], p. 86 (in English).⁸ The local variants are:

$$\rho \frac{d\mathbf{p}}{dt} = \nabla \cdot \boldsymbol{\sigma} + \rho (\mathbf{f} + \chi_B \cdot \boldsymbol{\omega}), \quad \mathbf{e} \cdot [(\mathbf{v} - \mathbf{w}) \otimes \rho \mathbf{p} - \boldsymbol{\sigma}] = \mathbf{0}, \quad (16)$$

where the second equation holds true, if $\chi_B \equiv \mathbf{0}$.

\mathbf{f} denotes the specific body force, $\boldsymbol{\sigma}$ is the Cauchy stress tensor. The specific momentum, \mathbf{p} , was introduced in Equation (5)₁. It should be pointed out that according to our nomenclature the body forces \mathbf{f} are a controllable supply and not a production term. Similarly, also the flux term with the traction $\mathbf{t} = \mathbf{n} \cdot \boldsymbol{\sigma}$ is controllable from outside. The term $\rho \chi_B \cdot \boldsymbol{\omega}$ accounts for a production of momentum due to structural transformations. It cannot be controlled from outside and was first proposed in [31]. By extending the way of speech of [49] or [51] momentum is conserved if there is no structural change so that the production χ_B vanishes. Also recall that sometimes Equation (15) is considered to be a **postulate** or **axiom**, which needs no further explanation. Other sources say that the left hand (kinematic) side of motion can be used to define the forces on the right, [61], p. 889, [57], p. 48.⁹ Maybe it is more prudent to say that forces are **primitive concepts**, which need no definition, see [33], p. 124: "Force, for Newton, was a concept given a priori, intuitively, and ultimately in analogy to human muscular force." The dilemma is even more clearly stated on p. 124 of the same reference: "The second law, likewise, has two possible interpretations: it may serve as a quantitative definition of force or as a generalization of empirical facts." As a compromise may it therefore suffice to say that forces and Newton's LEX II are manifestations of the **Aristotelian principle of cause and action**: The forces are the reason why we observe a change in motion, but we do not explain their origin. This is Western philosophy at its best.

- **Balances of angular momentum** For a generalized micropolar continuum the balance of angular momentum reads in global form,

$$\begin{aligned} \frac{d}{dt} \int_{V(t)} \rho (\mathbf{x} \times \mathbf{p} + \mathbf{s}) dV &= \oint_{\partial V(t)} \mathbf{n} \cdot (-\boldsymbol{\sigma} \times \mathbf{x} + \boldsymbol{\mu}) dA \\ &+ \int_{V(t)} \rho (\mathbf{x} \times (\mathbf{f} + \chi_B \cdot \boldsymbol{\omega}) + \mathbf{m} + \mathbf{v} \cdot \chi_B + \chi_J \cdot \boldsymbol{\omega}) dV. \end{aligned} \quad (17)$$

\mathbf{x} is the (current) position of the material particle,¹⁰ $\boldsymbol{\mu}$ is the couple stress tensor, and \mathbf{m} denotes the specific body couple. Note that the total specific angular momentum is the sum of the specific moment of momentum given by $\mathbf{x} \times \mathbf{p} = \mathbf{x} \times (\mathbf{v} + \mathbf{B} \cdot \boldsymbol{\omega})$ and the specific **dynamic spin** \mathbf{s} from Equation (5)₂. This is the total rotational momentum of a material micropolar point. The term $\mathbf{v} \cdot \chi_B + \chi_J \cdot \boldsymbol{\omega}$ accounts for a production of spin if structural transformations are present.

One could say that Equation (17) is the generalization of what is known as Euler's Second Law of Dynamics ([75], p. 132) for rigid bodies to generalized micropolar media. From the global equation the local equations in regular and singular points are readily obtained:

$$\rho \frac{d}{dt} (\mathbf{x} \times \mathbf{p} + \mathbf{s}) = \nabla \cdot (-\boldsymbol{\sigma} \times \mathbf{x} + \boldsymbol{\mu}) + \rho (\mathbf{x} \times (\mathbf{f} + \chi_B \cdot \boldsymbol{\omega}) + \mathbf{m} + \mathbf{v} \cdot \chi_B + \chi_J \cdot \boldsymbol{\omega}). \quad (18)$$

⁸ Also known as Euler's First Law of Dynamics ([75], p. 130).

⁹ The one-sidedness of this opinion is impressively analyzed in [58] where a whole section is dedicated to the topic "Arguments against the arguments for an impulse-only interpretation."

¹⁰ Therefore total angular momentum is introduced here w.r.t. the starting point of this position vector.

Angular momentum is conserved if there is no structural change so that the productions χ_B and χ_J vanish and then the following jump condition is valid:

$$\mathbf{e} \cdot [(\mathbf{v} - \mathbf{w}) \otimes \rho(\mathbf{x} \times \mathbf{p} + \mathbf{s}) + \boldsymbol{\sigma} \times \mathbf{x} - \boldsymbol{\mu}] = \mathbf{0}. \quad (19)$$

Note that the expressions for the temporal change of total angular momentum are **not** a consequence of the balances of momentum (15) or (16). On the continuum level the balance of angular momentum is a **postulate**: Similarly as the balance of momentum (15) introduces the concept of forces, the balance of angular momentum (17) puts forward moments in terms of the couple stress tensor $\boldsymbol{\mu}$ and the field of body couples, \mathbf{m} . Moreover, on the left hand “kinematic” side it introduces the new field called dynamic spin.

- **Balances of moment of momentum** In contrast to the balance of angular momentum the balance of moment of momentum and the balance of dynamic spin do not have the status of axioms. Rather they are consequences of the balances of momentum and angular momentum. In order to realize that we will stepwise eliminate the momentum parts in the balances for the angular momentum, Equations (17) and (18). To this end we start locally and multiply Equation (16) by $\mathbf{x} \times$. After several algebraic manipulations a balance for the moment of momentum is obtained in local form for regular points by purely mathematical reasoning,

$$\rho \frac{d}{dt}(\mathbf{x} \times \mathbf{p}) = -\nabla \cdot (\boldsymbol{\sigma} \times \mathbf{x}) - \boldsymbol{\sigma}_\times + \rho(\mathbf{x} \times (\mathbf{f} + \chi_B \cdot \boldsymbol{\omega}) + \mathbf{v} \times \mathbf{B} \cdot \boldsymbol{\omega}), \quad (20)$$

where $(\mathbf{a} \otimes \mathbf{b})_\times = \mathbf{a} \times \mathbf{b}$. By integrating these expressions over a volume we arrive at the balance of moment of momentum in global form,

$$\frac{d}{dt} \int_{V(t)} \rho \mathbf{x} \times \mathbf{p} dV = - \oint_{\partial V(t)} \mathbf{n} \cdot \boldsymbol{\sigma} \times \mathbf{x} dA + \int_{V(t)} [-\boldsymbol{\sigma}_\times + \rho(\mathbf{x} \times (\mathbf{f} + \chi_B \cdot \boldsymbol{\omega}) + \mathbf{v} \times \mathbf{B} \cdot \boldsymbol{\omega})] dV. \quad (21)$$

At this point several comments are in order:

- In view of the derivation of Equation (20) one may want to conclude that the balance of the moment of momentum is a consequence of the balance of momentum (16)₁. It is **not** independent nor **postulated**, rather it is derived from purely mathematical arguments. In fact, it is not really necessary to multiply the local balance of momentum (16)₁ by $\mathbf{x} \times$ and to perform various algebraic manipulations. Some may find it more straightforward to evaluate the temporal change of moment of momentum, i.e., of $\frac{d}{dt}(\mathbf{x} \times \mathbf{p})$ directly by, first, making use of the product rule and, second, by using the right hand side of (16)₁. The result (20)₁ and the follow-up conclusions are the same.
- According to the nomenclature established in context with production terms we must conclude that the balances of moment of momentum according to Equation (20)₁ or (21) are not conserved, even if we neglect all contributions from body forces and structural production terms. This is because of the term $\boldsymbol{\sigma}_\times$. Hence we could say that if the stress tensor is symmetric, the moment of momentum is conserved.
- In the same context it should be noted that the mechanics community rarely speaks of the conservation of moment of momentum. For example, Eringen refers to the requirement of a symmetric stress tensor in [11], p. 108 as the “second law of motion of Cauchy.” Sedov [60], p. 158 sees the balance of moment of momentum quite pragmatically as a tool: “Recall that earlier on four universal equations [Sedov refers to mass and momentum balance] were derived which describe the motion of a continuous medium. Now, these equation may be augmented by the three equations of angular momentum. In the classical case [Sedov means non-polar media], these three additional equations do not contain any new unknowns; they simply diminish the number of independent components of the stress tensor to six.” The same opinion is expressed in [23], p. 71. In view of that one may want to conclude that the requirement according to which classical, non-polar materials have a symmetric stress tensor is a consequence of the balance of moment of momentum. And then there is no further need to refer to the conservation of moment of momentum.
- **Balances of dynamic spin** Next we subtract the results for moment of momentum from the equations for total angular momentum (17) and (18), respectively. By doing so the local balances of dynamic spin remain:

$$\rho \frac{d\mathbf{s}}{dt} = \nabla \cdot \boldsymbol{\mu} + \boldsymbol{\sigma}_\times + \rho(\mathbf{m} + \mathbf{v} \cdot \chi_B + \chi_J \cdot \boldsymbol{\omega} - \mathbf{v} \times \mathbf{B} \cdot \boldsymbol{\omega}), \quad (22)$$

and the global counterpart:

$$\frac{d}{dt} \int_{V(t)} \rho \mathbf{s} dV = \oint_{\partial V(t)} \mathbf{n} \cdot \boldsymbol{\mu} dA + \int_{V(t)} [\boldsymbol{\sigma}_{\times} + \rho(\mathbf{m} + \mathbf{v} \cdot \boldsymbol{\chi}_B + \boldsymbol{\chi}_J \cdot \boldsymbol{\omega} - \mathbf{v} \times \mathbf{B} \cdot \boldsymbol{\omega})] dV. \quad (23)$$

Again, in the ductus of [49] or [51], dynamic spin is not conserved even if there is no structural change so that the productions $\boldsymbol{\chi}_B$ and $\boldsymbol{\chi}_J$ vanish. This is due to the non-symmetric stress tensor, which leads to the production $\boldsymbol{\sigma}_{\times} \neq \mathbf{0}$, which is the negative of the the corresponding term in the balance of moment of momentum. Note once more that this balance is not a direct consequence of the balance of translational momentum (16)₁. It follows *a posteriori* from the postulated balance of total angular momentum if the balance of moment of momentum is subtracted.

- **Balances of total energy** The balance for the total energy of a generalized continuum reads in global form

$$\begin{aligned} \frac{d}{dt} \int_{V(t)} \rho(\mathcal{E}_{\text{kin}} + u) dV &= \oint_{\partial V(t)} \mathbf{n} \cdot (\boldsymbol{\sigma} \cdot \mathbf{v} + \boldsymbol{\mu} \cdot \boldsymbol{\omega} - \mathbf{q}) dA + \\ &+ \int_{V(t)} \rho \left(\mathbf{f} \cdot \mathbf{v} + \mathbf{m} \cdot \boldsymbol{\omega} + \mathbf{v} \cdot \boldsymbol{\chi}_B \cdot \boldsymbol{\omega} + \frac{1}{2} \boldsymbol{\omega} \cdot \boldsymbol{\chi}_J \cdot \boldsymbol{\omega} + r \right) dV, \end{aligned} \quad (24)$$

and in local form

$$\rho \frac{d}{dt} (\mathcal{E}_{\text{kin}} + u) = \nabla \cdot (\boldsymbol{\sigma} \cdot \mathbf{v} + \boldsymbol{\mu} \cdot \boldsymbol{\omega} - \mathbf{q}) + \rho \left(\mathbf{f} \cdot \mathbf{v} + \mathbf{m} \cdot \boldsymbol{\omega} + \mathbf{v} \cdot \boldsymbol{\chi}_B \cdot \boldsymbol{\omega} + \frac{1}{2} \boldsymbol{\omega} \cdot \boldsymbol{\chi}_J \cdot \boldsymbol{\omega} + r \right), \quad (25)$$

where \mathbf{q} is the heat flux and r the volumetric heat supply. The total energy is conserved if there is no structural change so that the productions $\boldsymbol{\chi}_B$ and $\boldsymbol{\chi}_J$ vanish and then the following jump condition is valid:

$$\mathbf{e} \cdot \llbracket (\mathbf{v} - \mathbf{w}) \rho(\mathcal{E}_{\text{kin}} + u) - \boldsymbol{\sigma} \cdot \mathbf{v} - \boldsymbol{\mu} \cdot \boldsymbol{\omega} + \mathbf{q} \rrbracket = 0. \quad (26)$$

As such total energy conservation is also a **postulate** based on experimental evidence.

- **Balances of kinetic energy** Scalar multiplication of the local balance of momentum (16) by \mathbf{v} and of the local balance of spin (22) by $\boldsymbol{\omega}$, followed by algebraic manipulations leads to the local balance of **kinetic energy**:¹¹

$$\begin{aligned} \rho \frac{d\mathcal{E}_{\text{kin}}}{dt} &= \nabla \cdot (\boldsymbol{\sigma} \cdot \mathbf{v} + \boldsymbol{\mu} \cdot \boldsymbol{\omega}) - \boldsymbol{\sigma} : (\nabla \otimes \mathbf{v} + \mathbf{I} \times \boldsymbol{\omega}) - \boldsymbol{\mu} : \nabla \otimes \boldsymbol{\omega} \\ &+ \rho \left(\mathbf{f} \cdot \mathbf{v} + \mathbf{m} \cdot \boldsymbol{\omega} + \mathbf{v} \cdot \boldsymbol{\chi}_B \cdot \boldsymbol{\omega} + \frac{1}{2} \boldsymbol{\omega} \cdot \boldsymbol{\chi}_J \cdot \boldsymbol{\omega} \right), \end{aligned} \quad (27)$$

or after integration over the volume in global form:

$$\begin{aligned} \frac{d}{dt} \int_{V(t)} \rho \mathcal{E}_{\text{kin}} dV &= \oint_{\partial V(t)} \mathbf{n} \cdot (\boldsymbol{\sigma} \cdot \mathbf{v} + \boldsymbol{\mu} \cdot \boldsymbol{\omega}) dA - \int_{V(t)} (\boldsymbol{\sigma} : (\nabla \otimes \mathbf{v} + \mathbf{I} \times \boldsymbol{\omega}) + \boldsymbol{\mu} : \nabla \otimes \boldsymbol{\omega}) dV \\ &+ \int_{V(t)} \rho \left(\mathbf{f} \cdot \mathbf{v} + \mathbf{m} \cdot \boldsymbol{\omega} + \mathbf{v} \cdot \boldsymbol{\chi}_B \cdot \boldsymbol{\omega} + \frac{1}{2} \boldsymbol{\omega} \cdot \boldsymbol{\chi}_J \cdot \boldsymbol{\omega} \right) dV, \end{aligned} \quad (28)$$

where $(\mathbf{a} \otimes \mathbf{b}) : (\mathbf{c} \otimes \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d})$. The kinetic energy is **not conserved** even if there is no structural change so that the productions $\boldsymbol{\chi}_B$ and $\boldsymbol{\chi}_J$ vanish. This is due to the volumetric production terms related to the stress tensor and to the couple stress tensor.

- **Balances of internal energy** Subtraction of the balances of kinetic energy from the balances of total energy, Equations (24) and (25) results in the balance of internal energy (also known as the **First Law of Thermodynamics**) in global

¹¹ Alternatively one could have analyzed the temporal change of the specific kinetic energy directly. Also note that with a grain of salt the balance of kinetic energy may be considered as the scalar twin of the vector balance of the moment of momentum.

and in local form, respectively,

$$\frac{d}{dt} \int_{V(t)} \rho u \, dV = \int_{V(t)} (\boldsymbol{\sigma} : (\nabla \otimes \mathbf{v} + \mathbf{I} \times \boldsymbol{\omega}) + \boldsymbol{\mu} : \nabla \otimes \boldsymbol{\omega}) \, dV - \oint_{\partial V(t)} \mathbf{n} \cdot \mathbf{q} \, dA + \int_{V(t)} \rho r \, dV, \quad (29)$$

and

$$\rho \frac{du}{dt} = \boldsymbol{\sigma} : (\nabla \otimes \mathbf{v} + \mathbf{I} \times \boldsymbol{\omega}) + \boldsymbol{\mu} : \nabla \otimes \boldsymbol{\omega} - \nabla \cdot \mathbf{q} + \rho r. \quad (30)$$

Note that the internal energy does not contain production terms due to structural change. Nevertheless, it is not conserved due to the power terms that involve the stress tensor and the couple stress tensor. In our nomenclature these are volumetric production terms and not supplies.

2.2 | A survey of historical and recent literature

In his two papers [69,70] Truesdell questions after his derivation of the balance of angular momentum from a cloud of point masses the range of validity of the assumption that the internal forces are central. In fact he cites Joos [34] and Sommerfeld [63] as prime examples. Whilst Truesdell seems to accept this assumption for a cloud of mass points (“Gegen diesen Satz kann niemand Bedenken haben.”¹²) he finds its extrapolation to rigid bodies problematic and to the continuum definitely wrong.

Rightfully so! Surely a general continuum cannot simply be considered as a simple aggregate of point particles with radial forces of interaction. Here the intention is to model the behavior of more sophisticated matter within a material point or (more suggestively) Representative Volume Element (RVE). The RVE consists of molecules of high complexity the interaction between which will certainly be, in general, non-central. In fact, it was the crystal physicist Voigt who pointed this out explicitly, maybe for the first time, at the beginning of [73]: “Wir denken uns das homogene krystallinische Medium bestehend aus einem System von Molekülen, welche durch ihre Wechselwirkungen einander im Gleichgewicht halten.... Diese Wechselwirkungen sind Kräfte und Drehungsmomente, deren Componenten in unbekannter Weise mit der relativen Lage der Moleküle variiren.”¹³ The emphasis is on the word “Drehungsmomente” (turning moments), something that does not exist if the molecules were acting like point masses. And, consequently, after homogenization, this atomistic effect might enter the continuum scale.

But what about a rigid body, which is a special type of continuum, such that its constituting mass containing elements, which we may call “points” in the continuum sense, must, by definition, always keep a fixed distance to each other? Of course this is only an idealizing model of real matter. In elementary text books it is said that in reality a rigid body is a good model for very stiff materials, for example ceramics. Of course, the molecules that constitute the sintered grains of a ceramic are as complex as Voigt had them in mind. Hence, assuming a purely radial interaction seems faulty from the very start, unless one defines a rigid body in a very artificial limit as follows: Imagine a cloud of point masses, where each mass is connected to three next neighbors by rigid straight massless rods that are not all situated within the same plane. Then let the lengths of all these rods approach zero, such all mass points are “glued together” and “smeared out.” Then there is no reason to say that the law of moment of momentum cannot be derived since all the assumptions in context with a cloud containing a finite number of mass points stay valid. However, most likely everybody agrees that this limit process is rather far-fetched. But it seems this is how physicists were (and still are) taught.

Consider for example the early physics encyclopedia written by Thomson (Lord Kelvin) and Tait [68]. We read: “265. Every rigid body may be imagined to be divided into infinitely small parts. Now, in whatever form we may eventually find a *physical* explanation of the origin of the forces which act between these parts, it is certain that each such small part may be considered to be held in its position relatively to the others by mutual forces in lines joining them.” (note that these lines are nothing else but the infinitely small massless rigid rods connecting centers of mass elements mentioned above) and “266. From this we have, as immediate consequences of the second and third laws, and of the preceding theorems related

¹² “Against this sentence nobody can have reservations.”

¹³ “We imagine the homogeneous crystalline medium to consist of a system of molecules, which are in equilibrium with each other due to their interactions.... These interactions are forces and turning moments, the components of which vary in an unknown way with the relative position of the molecules.”

to Centre of Inertia and Moment of Momentum, a number of important propositions such as the following: ... (d) The rate of increase of moment of momentum, when the body is acted on by external forces, is the sum of the moments of these forces about the axis.” It is not surprising that such authoritative wording remained in the brains of physics generations to come. If we believe this lore, and many physicists do, the balance of moment of momentum is derivable from Newton’s second and third law, at least for rigid bodies.

However, what is questionable in Truesdell’s line of arguments, is to leave the reader under the impression that the great mechanics scholars of the past, in particular Euler, had all of these critical thoughts in mind, when the two sets of equations of motion were finally stated for the first time. Moreover, Newton is blamed by many in the mechanics community that he was quite ignorant about the concept of moment of momentum or “angular momentum,” in imprecise wording. Such statements are totally unfair to say the least. Indeed, Newton knew linear momentum, which he applied in combination with an immaterial lever arm, and dynamic equilibrium of forces between his gravitational (central) force and the centripetal fictitious force to provide a theoretical basis for Kepler’s laws of planetary motion. The latter concerns in particular Kepler’s second law, the equal area law, which is a manifestation of the conservation of angular momentum for a point mass subjected to a central force. All the details regarding this topic can be found in Newton’s original letter “De Motu Corporum in Gyrum”¹⁴ to Halley from 1684 provided in [21], p. 257, which entered the Principia [37] as explained in Chandrasekhar’s book [2] on p. 67.

Another cherished belief in the mechanics community is that Newton thought only in terms of point masses. Far from it! Indeed, Newton always thought holistically. For example if we turn to LEX 1 in the Principia [37], we read: “Trochus, cujus partes cohærendo perpetuo retrahunt sese a motibus rectilineis, non cessat rotari, nisi quatenus ab aëre retardatur.”¹⁵

However, it is true that he does not support his arguments by calculus in the Principia. Indeed, Newton argues geometrically even though calculus was invented by him. Chandrasekhar comments on these issues as follows: “J. E. Littlewood has conjectured (though I do not share in the conjecture—see below) that Newton had perhaps first constructed a proof based on calculus which ‘we can infer with some possibility what the proof was.’”; “As Littlewood has remarked, this construction ‘must have left its readers in helpless wonder.’” and “Littlewood was, of course, repeating a legend that has often been told, namely, that Newton first constructed proofs of most (if not all) of his propositions by calculus and then transliterated them into ‘his’ geometrical language. I do not believe in this legend. First, there are enough propositions that are proved directly by integral calculus, for example, Propositions XXXIX and XLI: and in Proposition LXXI he simply preferred not to give the proofs by calculus. Second, his physical and geometrical insights were so penetrating that the proofs emerged whole in his mind: ‘he was happy in his thoughts’ (qualifying de Morgan). Besides, where was the time to dissimulate? For my part, I am not surprised that ‘to the Newton of 1685’ the geometrical construction ‘that must have left its readers in helpless wonder’ came quite naturally (see the comments at the end of the next proposition).” from [2], p. 270. A good example of Newton’s way of presenting things, which is also in context with extended bodies, is his method of channels. He uses it to predict the flattening of planets, such as Jupiter, which he considers as fluidic, under centrifugal forces, cf., [37], p. 592.

Truesdell acknowledges Newton’s attempts at studying extended bodies, albeit somewhat reluctantly. For example after quoting the spinning top comment after LEX I he says: “It would be interesting to see if among Newton’s papers there is anything mathematical concerning the motion of rigid bodies.” Such a paper was found later. It is Manuscript V, The Laws of Motion Paper, documented in [21]. In its § 8. Newton discusses the fixation of the axis of a spinning body unless it is subjected to external moments. But Truesdell only laconically remarks “I do not find in § 8 of Manuscript V good grounds for such sweeping praise” and continues to shred Newton’s arguments to little pieces. In this context one is tempted to quote St. Mark 4:9, “Qui habet aures audiendi audiat.”¹⁶

We have emphasized that physicists tend to believe in Thomson–Tait’s idea that for a rigid body the law of moment of momentum can be derived from Newton’s second law if radial internal interaction forces are exclusively present. What about Euler, the ultimate idol of the mechanics community? If one reads his works objectively, Euler does something very similar, for example in his seminal paper on this subject [12], p. 223. Here at the utmost he motivates, definitely not postulates, and one is tempted to say that he **derives** the law of moment of momentum similarly to Thomson and Tait, with the exception that he considers mass elements instead of point masses, i.e., he takes the viewpoint of continuum theory, and rightfully, as we shall understand later, he does not mention internal interaction within a rigid body. Nevertheless

¹⁴ “On the motion of bodies in rotation.”

¹⁵ A top, whose parts by their cohesion are perpetually drawn aside from rectilinear motions, does not cease its rotation, otherwise than as it is retarded by air. [56], p. 83.

¹⁶ “He that hath ears to hear, let him hear.”

he indirectly uses Newton's second law in order to relate inertial forces of the body to moments applied from external sources. In his very own words: “§. 28. Cum igitur elemento dM , quod in puncto z concipimus, primo applicata sit vis $= dM(\frac{ddx}{dt^2})$ secundum directionem IA agens, ex ea nullum nascitur momentum pro hoc axe; pro axe autem IB nascetur momentum $= zdM(\frac{ddx}{dt^2})$ et pro axe IC momentum $= ydM(\frac{ddx}{dt^2})$ ”¹⁷ ... Hinc igitur pro quolibet axe habemus bina momenta elementaria, quae in partes contrarias vergunt; unde pro axe IA summa omnium momentorum elementarium erit

$$+ \int z dM \left(\frac{ddy}{dt^2} \right) - \int y dM \left(\frac{ddz}{dt^2} \right) = iS. \quad (31)$$

Eodem modo pro axe IB obtinebimus ...”¹⁸

And after that he states in §. 29 of this work his six equations for the linear and angular momentum for the first time in mechanics: “Hac igitur ratione sex nacti sumus aequationes, quas hic coniunctim conspectui exponamus”¹⁹

$$\begin{aligned} \text{I. } \int dM \left(\frac{ddx}{dt^2} \right) &= iP & \text{IV. } \int z dM \left(\frac{ddy}{dt^2} \right) - \int y dM \left(\frac{ddz}{dt^2} \right) &= iS \\ \text{II. } \int dM \left(\frac{ddy}{dt^2} \right) &= iQ & \text{V. } \int x dM \left(\frac{ddz}{dt^2} \right) - \int z dM \left(\frac{ddx}{dt^2} \right) &= iT \\ \text{III. } \int dM \left(\frac{ddz}{dt^2} \right) &= iR & \text{VI. } \int y dM \left(\frac{ddx}{dt^2} \right) - \int z dM \left(\frac{ddy}{dt^2} \right) &= iU. \end{aligned} \quad (32)$$

Note that Euler rescales time in his equations, in other words he measures force components P, Q, R and external moment components S, T, U in relation to the time required for a falling height of gravitational mass in one second. He expresses this by the ominous letter i and says on p. 222/223:

- “Quod si nunc simili modo omnes vires, quibus corpus hoc tempore sollicitatur etiam secundum illas ternas directiones resoluantur, atque ex omnibus coniunctis pro directionibus IA, IB, IC vires oriantur P, Q, R, ...”;
- “Scilicet si g denotet altitudinem lapsus grauium uno minuto secundo, loco $2g$ autem scribamus litteram i , quoniam littera g iam tanquam functio temporis in calculum ingreditur ...”;
- “...quamobrem designemus ista momenta, quae ex omnibus viribus sollicitantibus pro ternis axisbus IA, IB, IC nascantur, litteris S, T, V, ita ut his quantibus per i multiplicatis summae omnium momentorum elementarium, quas singulae vires acceleratrices suppediant aequari debeant.”²⁰

In summary the following can be said:

- It is true that physicists **derive** the balance of moment of momentum for **point** mass systems under the assumption of central interaction from Newton's second and third law, which is inevitable if such an idealized case is considered. There is nothing wrong with this derivation. It is formally correct, although one may consider it as unphysical since true point particles do not seem to exist in nature. Also recall that the assumption of central force interactions in a crystal lattice will lead to a paradox of the independence of elasticity constants known as Cauchy relations, see [39]. This shows the limitations of such an assumption for a general continuum.

¹⁷ An obvious typo: t^2 ; moreover, also note another not so obvious typo in equation VI. of Euler's manuscript: The lever arm z must be replaced by x . Nobody ever mentions it, rather it is tacitly corrected, e.g., in [67], p. 30, and Euler's work is left unblemished.

¹⁸ “Therefore let, first, to this element dM , which we conceive in the point z , be applied the force $= dM(\frac{ddx}{dt^2})$, second, in the direction IA acting, from which no moment about this axis is born; but for the axis IB the moment $= zdM(\frac{ddx}{dt^2})$ is born and for the axis IC the moment $= ydM(\frac{ddx}{dt^2})$... Hence, therefore, for any axis we have two elementary moments, which are directed in different parts; from where for the axis IA the sum of all elementary moments will be ... (31). In the same way we obtain for the axis IB ...”

¹⁹ “With this reasoning six equations are obtained, which we present here shown together ... (32).”

²⁰ • “And if now in a similar way all forces the body is subjected to in that moment are resolved in these three directions, and all of them combined in the directions IA, IB, IC the forces P, Q, R will originate ...”

• “Of course, if g denotes the height of a fall in gravity during one second, we write instead of $2g$ the letter i , because the letter g has already entered the calculation as a function of time ...”

• “...for this reason we designate these moments, which arise from all of the forces acting on the three axes IA, IB, IC, by the letters S, T, V, such that these quantities when multiplied by i must equal the sum of all elementary moments, which are provided by singular accelerating forces.”

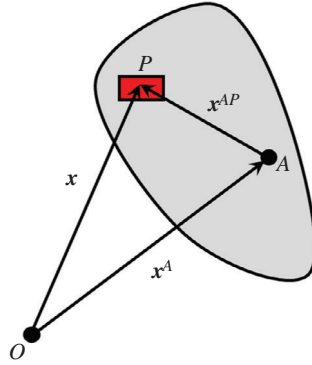


FIGURE 1 Nomenclature used in context with rigid bodies

- Extrapolating this kind of proof to a deformable continuum consisting on the atomistic level of complex matter is unwarranted. Truesdell's criticism is valid in full. It is necessary to **postulate** angular balances of momentum in the form of Equations (17) and (18) or the spin balances (23), (22) just as it was necessary to introduce the concept of internal energy in addition to kinetic energy.
- In his work Euler **motivates** the balance of moment of momentum for rigid bodies by connecting the lever action of inertial forces acting on mass elements to external moments, which are also forces \times lever arms for him.²¹ He does not address the issue of internal forces. Nowhere does he explicitly state that the balances of linear momentum and moment of momentum are independent. However, one may concede that he might have thought that both are independent and that an intelligent student of his works would simply have to read between the lines in order to realize it.

We now ask where is the rigid body hidden in our continuum equations, in particular in the equations for the momentum, (16), (15), the equations of total angular momentum, (17), (18), as well as in the relations for moment of momentum and for dynamic spin, (20)–(23)? The answer to this questions requires more space than offered by a bullet point, because, as among other things, the concept of a rigid body needs to be extended if micropolar media are concerned. Let us start with a briefing as to how the equations of motion for a rigid body are typically obtained in an Engineering Mechanics textbook (cf., [22], Chapters 20/21 or [18], Chapter 3). However, in preparation of our comparing analysis we will augment the arguments for beginners by making use of continuum mechanics terms. We also do that in order to put Euler's archaic mathematics into a modern context.

With reference to Figure 1 we initially note the following kinematic notions valid for rigid bodies: O denotes the footpoint of an inertial observer pointing with its position vector \mathbf{x} at an arbitrary material point P of the rigid body. A is an arbitrary fixed point on the rigid body, also known as the “pole,” the “pivot,” or the “base point” with the position vector \mathbf{x}^A . Finally, the vector \mathbf{x}^{AP} connects both points. It can rotate about A with an angular velocity $\boldsymbol{\omega}$, which is sometimes referred to as spin.²² However, its length cannot change because the body is rigid. Consequently Euler's kinematic relations for position, \mathbf{x} , translational velocity, $\mathbf{v} = \frac{d\mathbf{x}}{dt}$ and acceleration, $\mathbf{a} = \frac{d\mathbf{v}}{dt}$, hold:

$$\mathbf{x} = \mathbf{x}^A + \mathbf{x}^{AP} \Rightarrow \mathbf{v} = \mathbf{v}^A + \boldsymbol{\omega} \times \mathbf{x}^{AP} \Rightarrow \mathbf{a} = \mathbf{a}^A + \dot{\boldsymbol{\omega}} \times \mathbf{x}^{AP} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{x}^{AP}), \quad \dot{\boldsymbol{\omega}} = \frac{d\boldsymbol{\omega}}{dt}. \quad (33)$$

We now turn to the definition of total momentum, \mathbf{P} , and total moment of momentum, \mathbf{L} , of a rigid body. The mass element, $dm = \rho dV$ indicated in red in Figure 1 carries the momentum $\rho \mathbf{v} dV$ and the moment of momentum $\rho \mathbf{x} \times \mathbf{v} dV$. Hence it follows for the whole rigid body:

$$\mathbf{P} = \int_V \rho \mathbf{v} dV, \quad \mathbf{L} = \int_V \rho \mathbf{x} \times \mathbf{v} dV. \quad (34)$$

Note that in principle the mass density may depend on position \mathbf{x} but not explicitly on time: A rigid body can be heterogeneous. Moreover, by definition the volume V of a rigid body is no function of time. It is now worthwhile to

²¹ It is also interesting to note that no one mentions that Euler computes moments by $\mathbf{F} \times \mathbf{x}$ and not by $\mathbf{x} \times \mathbf{F}$.

²² Which must be distinguished from the dynamic spin fields $\mathbf{s} = \mathbf{J} \cdot \boldsymbol{\omega}$.

introduce two tensors of second rank, the moment of inertia tensor, $\mathbf{J}^{(A)}$ (symmetric) and the coupling tensor $\mathbf{B}^{(A)}$ (antisymmetric):

$$\mathbf{J}^{(A)} = \int_V \rho (\mathbf{x}^{AP} \cdot \mathbf{x}^{AP} \mathbf{I} - \mathbf{x}^{AP} \otimes \mathbf{x}^{AP}) dV, \quad \mathbf{B}^{(A)} = m(\mathbf{x}^A - \mathbf{x}^c) \times \mathbf{I}, \quad (35)$$

where $m = \int_V \rho dV$ is the total mass of the rigid body and the vector $\mathbf{x}^c = \frac{1}{m} \int_V \rho \mathbf{x} dV$ points to its center of mass. Note that $\mathbf{B}^{(A)}$ vanishes if A is chosen to coincide with the center of mass.

We can now rewrite (34):

$$\mathbf{P} = m\mathbf{v}^A + \mathbf{B}^{(A)} \cdot \boldsymbol{\omega}, \quad \mathbf{L} = \mathbf{x}^c \times (m\mathbf{v}^A) + (\mathbf{x}^A \times \mathbf{B}^{(A)} + \mathbf{J}^{(A)}) \cdot \boldsymbol{\omega}. \quad (36)$$

It is instructive to compare these two expressions, first, with the specific momentum of the micropolar material particle, $\mathbf{p} = \mathbf{v} + \mathbf{B} \cdot \boldsymbol{\omega}$, from Equation (15) and, second, with the specific angular momentum $\mathbf{x} \times \mathbf{p} + \mathbf{s} = \mathbf{x} \times (\mathbf{v} + \mathbf{B} \cdot \boldsymbol{\omega}) + \mathbf{v} \cdot \mathbf{B} + \mathbf{J} \cdot \boldsymbol{\omega}$ from (17). By observing that $\mathbf{x}^c \times (m\mathbf{v}^A) = \mathbf{x}^A \times (m\mathbf{v}^A) + \mathbf{v}^A \cdot \mathbf{B}^{(A)}$ the analogy is perfect. With the presented equations it is also easy to confirm that the total kinetic energy of a rigid body is given by:

$$E_{\text{kin}} = \frac{1}{2} m \mathbf{v}^A \cdot \mathbf{v}^A + \mathbf{v}^A \cdot \mathbf{B}^{(A)} \cdot \boldsymbol{\omega} + \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{J}^{(A)} \cdot \boldsymbol{\omega}, \quad (37)$$

which is built analogously to Equation (4). All of this shows how the idea of additional rotational degrees of freedom of a micropolar material point originated from rigid body kinematics.

Now we move on to the kinetics of the rigid body. We start from the following identities:

$$\frac{d\mathbf{P}}{dt} = \int_V \rho \mathbf{a} dV, \quad \frac{d\mathbf{L}}{dt} = \int_V \rho \mathbf{x} \times \mathbf{a} dV. \quad (38)$$

Note the following correspondences to Euler's notation,

$$\rho \mathbf{a} dV \rightarrow dM \left(\frac{d^2 x}{dt^2} \right), \dots, \rho \mathbf{x} \times \mathbf{a} dV \rightarrow +z dM \left(\frac{d^2 y}{dt^2} \right) - y dM \left(\frac{d^2 z}{dt^2} \right), \dots \quad (39)$$

Now we invoke Newton's Second Law and claim that mass \times acceleration=force. According to the principle of free body diagrams **all** forces on our mass element must be considered. This includes externally applied or "active" forces, such as gravity, $\rho \mathbf{f} dV = \rho \mathbf{g} dV$, \mathbf{g} being the gravitational acceleration, or point forces on the surface of the body, $\mathbf{t} dA = \sum_i \mathbf{F}^i \delta(\mathbf{x}^i) dA$, δ denoting the Dirac delta function.²³ However, reaction forces in between the rigid body particles must also be included. But if we assume that these "act radially" they will cancel out during integration w.r.t. all the material particles constituting the rigid body. They will also not contribute to moments, whereas gravity and point forces will, namely by the amounts $\rho \mathbf{x} \times \mathbf{f} dV = \rho \mathbf{x} \times \mathbf{g} dV$, and $\mathbf{x} \times \mathbf{t} dA = \sum_i \mathbf{x}^i \times \mathbf{F}^i \delta(\mathbf{x}^i) dA$, respectively. In summary, the momentum and the moment of momentum balances for the rigid body read (in combination with (33) for the left hand sides):

$$\frac{d}{dt} \int_V \rho \mathbf{v} dV = \int_V \rho \mathbf{f} dV + \oint_{\partial V} \mathbf{t} dA, \quad \frac{d}{dt} \int_V \rho \mathbf{x} \times \mathbf{v} dV = \oint_{\partial V} \mathbf{x} \times \mathbf{t} dA + \int_V \rho \mathbf{x} \times \mathbf{f} dV. \quad (40)$$

We are now in a position to compare this result with the global balances of total angular momentum (17), moment of momentum (21), and the dynamic spin (23). We start with the total angular momentum. Let us agree that if the material

²³ Actually the point forces, \mathbf{F}^i , could in principle be applied by some device to various particles at position \mathbf{x}^i within the body. However, according to the principle of line of action we can move them straight to the surface, which makes the practical "attachment" of the point force much easier.

particles of a rigid body are viewed as a classical, non-polar medium they cannot carry spin, $\mathbf{s} = \mathbf{0}$, the local angular velocity vanishes, $\boldsymbol{\omega} = \mathbf{0}$, they have no microinertia, $\mathbf{J} = \mathbf{0}$, there is no coupling tensor, $\mathbf{B} = \mathbf{0}$, the corresponding productions vanish, $\boldsymbol{\chi}_I = \mathbf{0}$, $\boldsymbol{\chi}_B = \mathbf{0}$, they have no couple stress tensor, $\boldsymbol{\mu} = \mathbf{0}$, and there are no specific body couples, $\mathbf{m} = \mathbf{0}$.²⁴ Then the total angular momentum balance (17) degenerates into the form (40)₂.

Let us now apply all of these rigid body assumptions in order to specialize the global balance of moment of momentum (21). It then becomes:

$$\frac{d}{dt} \int_{V(t)} \rho \mathbf{x} \times \mathbf{v} dV = \oint_{\partial V(t)} \mathbf{x} \times \mathbf{t} dA + \int_{V(t)} [-\boldsymbol{\sigma}_\times + \rho \mathbf{x} \times \mathbf{f}] dV. \quad (41)$$

Equation (41) teaches us that the moment of momentum of a rigid body can be changed by applying external moments through surface tractions and body forces, such as gravity. Moreover, Equation (41) contains the volumetric production term $\boldsymbol{\sigma}_\times$. This brings us back to the discussion in context with Equation (20): The purpose of this equation is to conclude either that moment of momentum is conserved if the stress tensor is symmetric or that the purpose of this equation is to require us to assume that the stress tensor of a classic material is symmetric. Moreover, if we use the theorem of Gauss–Ostrogradsky in combination with Cauchy’s relation $\mathbf{t} = \mathbf{n} \cdot \boldsymbol{\sigma}$ this equation shows us that there can be a state of stress inside a rigid body, albeit only a symmetric one. However, the deformation associated with this stress vanishes by definition. More mathematically oriented discussions on this issue can be found in [17] and [14].

Finally, under the assumptions made the spin balances (23) and (22) reduce to:

$$\boldsymbol{\sigma}_\times = \mathbf{0}, \quad (42)$$

or in words, the stress tensor of a non-polar material must be symmetric.

This concludes the rigid body model of a classic material. But indeed, there have been attempts to extend the concept of a rigid body to micropolar materials. This is known as the concept of a “quasi-rigid body.” In [77] we find on p. 246 the following revealing statement: “Квазитвердое тело можно представить себе следующим образом. Пусть дано абсолютно твердое тело, которое будем называть несущим. Для наглядности можно представлять себе несущее тело в виде тела, в котором имеется множество маленьких полостей. Пусть в каждой полости установлен миниатюрный гироскоп, центр масс которого неподвижен относительно несущего тела. Гироскоп состоит из вращающегося ротора, закрепленного в специальной конструкции, называемой кардановым подвесом. При вращении ротора его ось может поворачиваться. Если ось ротора закреплена относительно несущего тела, а сам ротор является телом вращения, то распределение массы в таком квазитвердом теле не меняется в процессе движения. Квазитвердое тело, распределение массы в котором не меняется в процессе движения, называется гириостатом. Если квазитвердое тело состоит из односпиновых частиц, то более точным образом квазитвердого тела является кристаллическая решетка (безынерционное несущее тело), в узлах которой находятся быстровращающиеся атомы. На самом деле атом нужно моделировать многоспиновой частицей, но в иллюстративном примере можно ограничиться и односпиновыми атомами, которые в Природе не встречаются.”²⁵

This is an interesting idea, indeed, because it kindles hope to supplement the non-intuitive abstract notions of quantum mechanics by something “more tangible.”

²⁴ Electromagnetic forces cannot be part of the “purely mechanical” rigid body presented here. Indeed, often moment couples \mathbf{m} are said to be induced by electro-magnetic action in terms of polarization and magnetization. In a strictly rigid body such effects based on separation and induction of charges are not allowed. They do not fit into the scheme presented here. Moreover, “screwdriver moments” which are the epitome of \mathbf{m} in elementary mechanics textbooks must be resolved into a couple of applied forces separated by a perpendicular distance. They are then part of the volume or of the surface term of the balance of moment of momentum shown in (40) depending on where they act.

²⁵ “A quasi-rigid body can be imagined as follows. Let an absolutely rigid body be given, which we shall call the carrier. For clarity, one can imagine a carrier body in the form of a body in which there are many small cavities. Let a miniature gyroscope be installed in each cavity, the center of mass of which is stationary relative to the supporting body. The gyroscope consists of a rotating rotor fixed in a special design called a gimbal. When the rotor rotates, its axis can rotate. If the rotor axis is fixed relative to the supporting body, and the rotor itself is a body of revolution, then the mass distribution in such a quasi-solid does not change during movement. A quasi-rigid body, the mass distribution of which does not change during movement, is called a *gyrostat*. If a quasi-solid body consists of single-spin particles, then in a more accurate way a quasi-solid body is a crystal lattice (inertialess carrier body), in the nodes of which there are rapidly rotating atoms. In fact, an atom must be modeled by a multispin particle, but in an illustrative example, one can limit oneself to single-spin atoms, which are not found in nature.”

TABLE 1 Nomenclature of angular momentum and related notions in mechanics

Ref.	Independent notions	Total angular momentum	Moment of momentum	Dynamic spin
[71]	Yes, Sect. 196	Unclear	= Total angular momentum, p. 482, 531	No
[19]	No, Sect. 84	No	= Total angular momentum	No
[10]	Yes, Sect. 1.4	Yes, p. 13	= Total angular momentum	Yes, p. 13
[7] ^a	Yes, p. 47	Hidden in Sect 1.10 and p. 47	= Total angular momentum	Yes, p. 33, 47
[74]	No, Chapter 8	= Moment of momentum	No	No
[22]	No, Sect. 15.6, 21.4	= Moment of momentum	No	No
[18]	No, Sect. 3.2	= Moment of momentum	No	No
[5]	Yes, Sect. 3.2	Yes, p. 17	= Total angular momentum	Simplified, p. 17

^a[8] and [9] are repetitions.

TABLE 2 Opinion about angular momentum and related notions in physics

Ref.	“Derived” by using point masses	Uncritically applied to rigid bodies	Angular momentum	Spin
[13]	Yes, Section 18.4	Yes	= Moment of momentum	No
[35]	Yes, Chapter 8, p. 240	Yes	= Moment of momentum	No
[38]	Yes, § 9, § 31, § 33	Yes	= Moment of momentum	No
[16]	Yes, Chapter 5, p. 240	Yes	= Moment of momentum	No
[23]	No	No	Yes, Section 2.3.3, p. 69	Yes
[40]	Yes, Section 2.2, p. 77 and p. 85	Yes	= Moment of momentum	No

2.3 | Angular momentum in mechanics and physics education

In Table 1 we investigate in chronological order the nomenclature and the distinctions that are made in some relevant mechanics related monographs and textbooks, namely Truesdell/Toupin 1960 [71], Hamel 1967 [19], Eringen 1976 [10], 1999 [7], and Zhilin 2003 [74], Hibbeler 2010 [22], Gross et al. 2011 [18], and Eremeyev et al. 2012 [5]. It should be noted that the list does not claim to be comprehensive. It merely serves as an illustration of the general situation. By the “independent notions” we mean that a distinction is made between total angular momentum $\mathbf{x} \times \mathbf{v} + \mathbf{s}$, which comprises the moment of momentum $\mathbf{x} \times \mathbf{v}$, and the dynamic spin **field** \mathbf{s} separately. The following observations are made:

- University textbooks on Engineering Mechanics do not mention the notion of a dynamic spin field because they deal exclusively with the dynamics of rigid bodies related to non-polar materials.
- They introduce and illustrate the moment of momentum principle for rigid bodies by following Euler’s “derivation.” Nevertheless, they also tend to stress its independence from Newton’s LEX II by elevating it to the rank of a “Principle.” Sometimes it is postulated as independent and both are distinguished as Euler’s First and Second Law.
- Monographs dedicated to higher continua and to micropolar media make a strict distinction between the various forms of “angular momentum.” They emphasize the notion of dynamic spin, its novelty and distinction from moment of momentum, at least in some simplified form.
- In conclusion it is fair to say that the mechanics community is aware of the problem and actively deals with it.

The situation in physics education is illustrated in Table 2 by some selected textbook examples, some of which are very famous. The screening test was also performed in chronological order: The Feynman Lectures on Physics 1963 [13], Berkeley Physics Course 1973 [35], Landau Lifshitz revised edition 1993 [38], Goldstein revised edition 2002 [16], Hutter and Joenhnk 2004 [23], and quite recent Lindner and Strauch 2018 [40]. Indeed,

- the presentation of angular momentum is in a very sad state of affairs. With one exception no breakdown into moment of momentum and dynamic spin is made in these books.

- Even the most prominent textbooks of physics (all of them published after Truesdell's papers, partly in revised and enlarged editions) derive the balance of moment of momentum from clouds of point masses assuming radial interaction forces. Nothing substantial has changed since Thomson and Tait [68].
- There is simply no awareness for the problematic nature of angular momentum.

Why is this so? One reason could be that today's physicists are too caught up in quantum mechanics and consider mechanics as a dead science. Surely, quantum particles exhibit something which is also referred to as spin and angular momentum but nobody derives that in terms of "mass times lever arm arguments" so it seems. The time seems ready to free oneself from prejudice and to connect both worlds, at least by mechanical analogies.

2.4 | Example I: Force and moment-free curvilinear motion of a body point

Section 3.2 of [77] presents a most intriguing application of the coupling tensor \mathbf{B} from Equation (4). Zhilin specializes the equations from Section 2.1, which are valid for the continuum, to the case of a point particle that carries mass m **and** rotational inertia. Because this is not the normal point particle we are used to in mechanics Zhilin calls this entity a тело точка, a "body point," in order to emphasize that it can carry also rotational properties.²⁶ For convenience the coupling tensor is assumed to be isotropic just as the microinertia, \mathbf{J} :

$$\mathbf{B} = B\mathbf{I}, \quad \mathbf{J} = J\mathbf{I}, \quad (43)$$

where B and J are constants. As initial conditions this body point is assigned a constant translational speed, \mathbf{v}_0 and a constant angular velocity, $\boldsymbol{\omega}_0$. There are no external forces and moments, no stress tensor, no couple stress tensor, and no structural production terms. Hence linear and angular momentum are conserved and the right hand sides of Equations (16)₁ and (18)₁ vanish completely. After uncoupling and solving the remaining ordinary differential equations the angular velocity, $\boldsymbol{\omega}$, develops according to:²⁷

$$\boldsymbol{\omega}(t) = [1 - \cos(\alpha t)]\mathbf{n} \cdot \boldsymbol{\omega}_0 \mathbf{n} + \cos(\alpha t)\boldsymbol{\omega}_0 + \sin(\alpha t)\mathbf{n} \times \boldsymbol{\omega}_0, \quad (44)$$

where

$$\alpha = \frac{Ba}{B^2 - J} \quad \text{and} \quad \mathbf{v}_0 + B\boldsymbol{\omega}_0 = \mathbf{a} = a\mathbf{n}, \quad (45)$$

\mathbf{n} being a unit vector indicating a direction.

The translational speed and the path of the particle are given by

$$\mathbf{v}(t) = \mathbf{v}_0 + B[(1 - \cos(\alpha t))(\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) - \sin(\alpha t)\mathbf{n} \times \mathbf{I}] \cdot \boldsymbol{\omega}_0 \quad (46)$$

and

$$\mathbf{x}(t) = \mathbf{x}_0 + \mathbf{v}_0 t + B\left[\left(t - \frac{1}{\alpha} \sin(\alpha t)\right)(\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) - \frac{1}{\alpha}(1 - \cos(\alpha t))\mathbf{n} \times \mathbf{I}\right] \cdot \boldsymbol{\omega}_0, \quad (47)$$

respectively.

The first two terms on the right hand side are the classical result according to Newton's LEX I, cf., [37]: "Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur status illum mutare."²⁸ Clearly, the terms on the second line show the difference between a "Newtonian" and a "Eulerian mechanics." The latter possesses the means of a balance of angular momentum including dynamic spin and a coupling between translational and angular parts of the kinetic energy: The mass point moves along a curved path. However, note

²⁶ The ordinary Newtonian point mass is called *материальная точка* in Russian, which could easily be confused with the continuum mechanics term "material point."

²⁷ A derivation of these results is given in the Appendix.

²⁸ Every body perseveres in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon. [56], p. 86.

that all of these differences to the classical point of view arise only if we allow the coupling coefficient B to be different from zero.

At this stage several comments are in order:

- It is fair to say that we just got very much used to the modeling concept according to which a point can carry translational inertia, m , and we do not question this any more. However, the question must be raised how a mass point can carry rotational inertia in terms of \mathbf{J} and \mathbf{B} or, more precisely, in terms of kinetic energies $\frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{J} \cdot \boldsymbol{\omega}$ and $\mathbf{v} \cdot \mathbf{B} \cdot \boldsymbol{\omega}$.
- It is known from traditional electrodynamics that a charged point particle moving with the velocity \mathbf{v} moves along a helical line if subjected to the Lorentz force $q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. It is for this reason why Zhilin also refers to B as electric charge q in [75], p.73: "...параметр q определяет некое новое свойство частицы, которое условно будем называть зарядом."²⁹
- Is the notion of an electrically charged point particle substantiated by experiment? As far as an electron is concerned it is assumed that based on current experimental accuracy its size must be smaller than 10^{-19} m. It can therefore considered to be the ultimate representation of a charged point particle in reality. Indeed, in classical physics terminology this object carries mass, electric charge, and what is more, intrinsic quantum spin. Of course, we do not want to go so far to say that a free electron moves in a predictive manner along a helix according to the theory presented above. At most the presented analysis provides a mechanical analogue to charge in terms of the \mathbf{B} tensor. However, mechanical analogies are important, because we know easier how to improve and how to deal with them.
- The example has two aspects. First, there is the coupling tensor \mathbf{B} , which is an additional degree of freedom. It originally stems from rigid body dynamics but there it is an antisymmetric tensor, which vanishes if the pole and the center of mass coincide. Here it is a completely new independent variable. Second, the example also shows the importance of a fully independent balance of dynamic spin, which is then coupled to translational momentum. It offers, under the absence of forces and moments, unexpectedly the possibility to predict curved motion, which otherwise is possible only by using the concept of a Lorentz force acting on a moving charged particle. Slightly bolder we may want to say, "an electron behaves as if it were a тело точка."
- As it was explained above 2x3 equilibrium equations of motion were presented by Euler in [12], three for the linear momentum and three for the moment of momentum (not for the total angular momentum). Euler did this for practical reasons, namely for studying the dynamics of three-dimensional rigid bodies. Surely Euler did not study mass points carrying rotational inertia. This idea is a recent one and its purpose is to enable us to model materials with higher internal degrees of freedom. However, this also means that we must always strive for a physics based explanation and avoid superficial application of formal mathematics.

2.5 | Example II: The æther, a micropolar medium?

Let us present and investigate Maxwell's equations from the viewpoint of a certain school of rational continuum theory. To this end we refer the reader mostly to Toupin's Chapter F. in [71], p. 660. Moreover, the following (more recent) references are most useful in this context: Chapter 9 of [49], p. 304, [36], Section 2.2 of [24], p. 9, [65], or Chapter 13 of [52]. It should be pointed out that there are other continuum approaches to electromagnetism. However, this paper is not the forum to discuss the differences. We will focus on local forms of Maxwell's equations in regular points. The intention of this section is to show that their structure can formally be "derived" either from the equations of non-polar media with an antisymmetric stress tensor or from the local balances in regular points for micropolar media, *i.e.*, they can be motivated mechanically. Recall the following:

- Faraday's law (the law of electromotoric force):

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = \mathbf{0} \quad (48)$$

- Conservation of magnetic flux (absence of magnetic monopoles):

$$\nabla \cdot \mathbf{B} = 0 \quad (49)$$

²⁹ "...the parameter q defines a certain new property of the particle, which, on probation, we shall call the charge."

- Ørsted-Ampère law (conservation of total charge, q , and total electric current, \mathbf{j} ³⁰):

$$-\frac{\partial \mathbf{D}}{\partial t} + \nabla \times \mathbf{H} = \mathbf{j} + q\mathbf{v} \quad (50)$$

- Gauss' law (source of electric charge):

$$\nabla \cdot \mathbf{D} = q \quad (51)$$

The symbols have the following meaning: \mathbf{E} and \mathbf{B} are force fields inherent to the electromagnetic Lorentz force. They are called the electric and the magnetic field, respectively. \mathbf{D} and \mathbf{H} are potentials that were introduced formally in order to satisfy the conservation law for electric charge and electric current identically,

$$\frac{\partial q}{\partial t} + \nabla \cdot (q\mathbf{v} + \mathbf{j}) = 0 \quad (52)$$

as can be checked easily by inserting Equations (50) and (51). \mathbf{D} and \mathbf{H} are known as total charge and total electric current potentials, respectively. Moreover, q denotes the total electric charge density and \mathbf{j} is the total electric current vector. In the form shown above the Maxwell equations hold in all systems for all type of observers. However, there are simple proportionality relations between the force fields, \mathbf{E} and \mathbf{B} , and the potentials, \mathbf{D} and \mathbf{H} , which are valid in this form only in an inertial frame of reference, also known as a Lorentz system, or, if we so wish, in the æther system. These are the Maxwell–Lorentz–æther relations, which read:

$$\mathbf{D} = \varepsilon_0 \mathbf{E}, \quad \mathbf{H} = \frac{1}{\mu_0} \mathbf{B} \quad (53)$$

where $\varepsilon_0 = 8.85 \times 10^{-12} \frac{\text{As}}{\text{Vm}}$ and $\mu_0 = 4\pi \times 10^{-7} \frac{\text{Vs}}{\text{Am}}$ are the dielectric constant and the permeability of the vacuum,³¹ which combined yield the speed of light constant, $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$. Recall that if we insert the Maxwell–Lorentz æther relations into Equations (48)–(51) and define vacuum such that it is a space free of charges and electric currents, wave equations for the electric and for the magnetic fields are obtained:

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} = c^2 \Delta \mathbf{E}, \quad \frac{\partial^2 \mathbf{B}}{\partial t^2} = c^2 \Delta \mathbf{B}. \quad (54)$$

These equations reflect the propagation of electromagnetic waves at a finite speed, c , through vacuum. It is today's common belief and knowledge that these waves do not need a material medium to travel within. This was rather different in the days of James Clerk Maxwell. Indeed he says in [48], p. 572: “Whatever difficulties we may have in forming a consistent idea of the constitution of the æther, there can be no doubt that the interplanetary and interstellar spaces are not empty, by a material substance or body, which is certainly the largest, and probably the most uniform body of which we have any knowledge.” This was a very strong belief of his and, indeed, he went through a lot of trouble to conceive mechanical models for the æther. Probably the most important of his publications in this context are first a series of papers, namely [43–46]. Here he presents the idea of “molecular vortices” for a mechanically-based understanding of electrodynamic phenomena. Second, we must mention [47], where he presented his equations for the first time in Part III, p. 480 of this publication. However, it is fair to say that only a hodge-podge of formulae is shown, where the balances of the four electromagnetic fields are mixed with simple linear constitutive equations, for example Ohm's law. Moreover, vector notation is not used. Rather a kaleidoscope of different letters for each of the components of the various fields is shown, and no distinction is made between partial and total derivatives in space and time. In short, it is slightly difficult to comprehend. One of the first to provide a proper vector notation of Maxwell's equations as we know it was Heaviside (see, for example, [20], p. 429). The mechanical interpretation of the Maxwell equations did not remain unopposed, because it suffered from internal

³⁰ Note that the electric current \mathbf{j} is a non-convective transport of charge similar to the heat flux, which can be considered as a non-convective transport of internal energy. On a microscale it can be linked to the movement of microscopic charges. However, on the continuum scale this movement is not visible. It takes a homogenization procedure to obtain \mathbf{j} from processes happening on the microscale, see [4] for more details.

³¹ Or of the æther we may want to say.

contradictions. Moreover, the material existence of the æther could not be unambiguously be verified, and many say that it was falsified in the famous Michelson–Morley experiment. Nevertheless, mechanical analogies are always helpful to understand something for which we have no hereditary feeling. In this spirit the following shall be said.

In [64] Sommerfeld made an attempt to connect electromagnetism with the equations of motion for a linear-elastic non-polar continuum. We quote from p. 96: “Bekanntlich hat man im 19. Jahrhundert der Optik einen Lichtäther zugrunde gelegt, der nach Möglichkeit die Eigenschaft eines gewöhnlichen elastischen Körpers haben sollte. Aber schon bei dem einfachsten Problem der Reflexion und Brechung traten Schwierigkeiten auf, ... Diese veranlaßten MACCULLAGH schon 1839 dazu, den Anschluß an die gewöhnliche Elastizitätstheorie aufzugeben und eine von diesen Schwierigkeiten freie Darstellung der Optik zu entwickeln, die, wie sich später zeigte, mit der elektromagnetischen Optik von MAXWELL (1864), ..., formelmäßig übereinstimmt. Was wir im folgenden bringen werden, ist eine Ausdeutung der MACCULLAGHschen Gleichungen ...”³² Then he begins to “derive” Maxwell’s equations based on a study of the torque and corresponding twist of a volume element made of quasilinear-elastic material, but with an anti-symmetric stress tensor. He inserts the result in the balance of momentum, specifically in the divergence term of the stress tensor. Moreover he linearizes and neglects the convective part of the inertial term. The latter he motivates by specializing to slow motion (“... sehen seine Bewegung als langsam an ...”³³). Finally he arrives at:³⁴

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\frac{k}{2} \nabla \times \boldsymbol{\varphi}, \quad (55)$$

where the expression $\boldsymbol{\sigma} = -\frac{k}{2}(\mathbf{I} \times \boldsymbol{\varphi})$ was chosen for the stress tensor, k is the “twist modulus,” and $\boldsymbol{\varphi}$ is the twist angle vector. He then adds the kinematic constraint between twist angle and the vorticity,

$$\frac{\partial \boldsymbol{\varphi}}{\partial t} = \frac{1}{2} \nabla \times \mathbf{v}. \quad (56)$$

All of this is complemented by interesting comments, such as, “... wenn wir auch hier $\frac{d\boldsymbol{\varphi}}{dt}$ mit $\frac{\partial \boldsymbol{\varphi}}{\partial t}$ vertauschen ...,” and “wir setzen noch Inkompressibilität voraus und fügen die aus der Bedeutung von $\boldsymbol{\varphi}$ als Rotation des Verschiebungsvektors folgende Bedingung hinzu:”³⁵

$$\nabla \cdot \mathbf{v} = 0, \quad \nabla \cdot \boldsymbol{\varphi} = 0. \quad (57)$$

Now assignments between mechanical and electrical quantities are made. They depend on factors of proportionality, α and β , respectively, “die ebenso wie die Vorzeichen von der Wahl der Einheit abhängen, in denen \mathfrak{E} und \mathfrak{H} gemessen werden, sowie von der Vorzeichenwahl bei der elektrischen Ladung und der magnetischen Polstärke”³⁶:

$$\begin{aligned} \text{(a)} \quad \mathbf{v} &= \pm \alpha \mathbf{E}, \quad \boldsymbol{\varphi} = \mp \beta \mathbf{H}, \\ \text{(b)} \quad \mathbf{v} &= \pm \alpha \mathbf{H}, \quad \boldsymbol{\varphi} = \pm \beta \mathbf{E}. \end{aligned} \quad (58)$$

³² Translation from [62], p. 108: “In 19th century physics, a material carrier was assumed for the optical phenomena, equipped as far as possible with the properties of ordinary elastic bodies. This construction, however, led to difficulties even in the most elementary problem of reflexion and refraction, ... As early as 1839 MacCullagh tried to drop the connection with the ordinary theory of elasticity with the aim to develop a representation of optics that would be free of the difficulties mentioned. It turned out later that his theory agreed formally with Maxwell’s electro-magnetic optics (1864), in particular as far as the optics of transparent bodies is concerned. The following remarks should be considered as an interpretation of MacCullagh’s equations.”

³³ In the English version [62] this statement is incorrectly translated “... consider its displacement as small ...”

³⁴ This equation and the assumption of an antisymmetric stress tensor should be viewed critically in context with the theory for linearly deformable micropolar solids (see [10], p. 27 or in more modern terminology [15]). Accordingly we must conclude that Sommerfeld assumes what is known today as perfect coupling, or a medium with constrained rotation. Then the microrotation $\boldsymbol{\varphi}$ and the macrorotation, $\frac{1}{2} \nabla \times \mathbf{u}$, \mathbf{u} being the displacement field, are equal. However, this is a degenerate case and Sommerfeld’s assumption of an antisymmetric stress tensor is problematic, since we have learned that if there is no dynamic spin (or microrotation) the stress tensor must be symmetric.

³⁵ In rather free translation from [62]: “where $\boldsymbol{\omega} = \frac{d\boldsymbol{\varphi}}{dt}$ was replaced by $\frac{\partial \boldsymbol{\varphi}}{\partial t}$. As a further assumption we add the incompressibility of the ether; we also note that $\boldsymbol{\varphi}$, being the curl of the displacement vector, has no divergence.”

³⁶ In translation from [62], p. 110: “that depend on the choice of units for E and H ” and “The signs in these equations depend on the signs chosen for the units of the electric charge and magnetic pole strength.”

Hence from the two choices the final result is obtained:

$$\begin{aligned}\frac{\partial \mathbf{D}}{\partial t} &= \nabla \times \mathbf{H}, \quad \nabla \cdot \mathbf{D} = 0, \\ \frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times \mathbf{E}, \quad \nabla \cdot \mathbf{B} = 0.\end{aligned}\tag{59}$$

These are the Maxwell equations for the absolute vacuum, *i.e.*, in an inertial frame without electric charges and without electric currents. To obtain them use has been made of the Maxwell–Lorentz æther relations (53) and the following relations:

$$\begin{aligned}\text{(a)} \quad \varepsilon_0 &= \frac{\rho}{k} \frac{2\alpha}{\beta}, \quad \mu_0 = \frac{2\beta}{\alpha}, \\ \text{(b)} \quad \mu_0 &= \frac{\rho}{k} \frac{2\alpha}{\beta}, \quad \varepsilon_0 = \frac{2\beta}{\alpha}.\end{aligned}\tag{60}$$

In both cases we obtain the same relation for the speed of light:

$$\varepsilon_0 \mu_0 = \frac{4\rho}{k} \Rightarrow c = \sqrt{\frac{k}{4\rho}}.\tag{61}$$

In summary we conclude that for choice (a)

- the momentum balance in combination with an antisymmetric stress tensor and a quasi-linear-elastic relation between stress and twist yield the Ørsted–Ampère law;
- incompressibility yields Gauss law;
- the vorticity condition yields Faraday’s law;
- and the fact that the twist stems from the curl of the displacement field leads to the non-existence of magnetic monopoles.

In the case of choice (b) we must say:

- the momentum balance in combination with an antisymmetric stress tensor and a quasi-linear-elastic relation between stress and twist yield Faraday’s law;
- incompressibility leads to the non-existence of magnetic monopoles;
- the vorticity condition yields the Ørsted–Ampère law;
- and the fact that the twist stems from the curl of the displacement field yields Gauss law.

This ambivalence is due to the high symmetry of the Maxwell equations if there are no electric charges and electric currents. We conclude this curious derivation with a quotation from Sommerfeld [64], p. 99: “Es liegt uns fern, diesem Äthermodell irgendeine physikalische Realität beizulegen. Man hat sich schon um die Jahrhundertwende überzeugt, daß alle Bemühungen um eine mechanische Erklärung der MAXWELLSchen Gleichungen zur Fruchtlosigkeit verurteilt sind. Es kann sich nicht um mechanische Erklärungen, sondern bestenfalls um mechanische Analogien handeln. Die MAXWELLSchen Gleichungen liegen dem elektrischen Aufbau der gewöhnlichen Materie zugrunde; man kann daher nicht erwarten, sie aus den Eigenschaften der ponderablen Körper zu erklären. Unsere Betrachtung hat aber vielleicht insofern ihre Berechtigung als sie zeigt: Wenn man ein Medium „Äther“ konstruieren will, das als Substrat der MAXWELLSchen Gleichungen dienen soll, so muß man ihm diametral entgegengesetzte Eigenschaften beilegen wie den gewöhnlichen Stoffen, nämlich eine *absolute Richtungsorientierung* gegen den Raum, nicht eine *relative Orientierung der Volumelemente gegeneinander*, wie sie dem elastischen Körper zukommt.”³⁷

³⁷ An extended translation with a typo from [62]: “It is by no means our intention to assign any physical reality to this “ether model”. Physicists had convinced themselves by the turn of the century that all attempts at a mechanical explanation of Maxwell’s equations were doomed to failure. What we mean here is not a mechanical explanation but, at best, a mechanical analogy. Maxwell’s equations are among the fundamentals of the electrical theory

So far Sommerfeld's attempts. Recall that his whole analysis stands and falls with the assumption of an antisymmetric stress tensor, a familiar feature of higher gradient or polar media. It seems therefore only natural to repeat such a motivation of Maxwell's equations but now from the standpoint of micropolar theory and the balances (complemented by suitable constitutive relations for the "micropolar æther") from Section 2.1. Such attempts have been made in the school of Zhilin in St. Petersburg. Unfortunately, these works are not so well known, also because some of the initial ones are written in Russian. Therefore, in what follows we repeat some of the arguments from the following references that present general remarks to the mechanics of electromagnetism and to micropolar modeling of the æther: [75], § 6.6, § 7.1, [76,77], § 12.3. However, more recently disciples of Zhilin endeavored to apply micropolar theory also for a better understanding of general electromagnetically susceptible materials. The interested reader is referred to [26–30].

We start with the spin balance (22)₁, by making the following assumptions:

$$\rho = \text{const.}, \quad \mathbf{s} = J\boldsymbol{\omega}, \quad \mathbf{v} = \mathbf{0}, \quad \boldsymbol{\sigma} = \mathbf{0}, \quad \boldsymbol{\chi}_B = \mathbf{0}, \quad \boldsymbol{\chi}_J = \mathbf{0}. \quad (62)$$

In words, the æther material is incompressible and isotropic w.r.t. its microinertia, it rests, does not carry stress, and shows no structural change. Then $\frac{d(\cdot)}{dt} = \frac{\partial(\cdot)}{\partial t} + \mathbf{v} \cdot \nabla(\cdot) \equiv \frac{\partial(\cdot)}{\partial t}$ ³⁸ and we obtain:

$$\frac{\partial \rho J \boldsymbol{\omega}}{\partial t} = \nabla \cdot \boldsymbol{\mu} + \rho \mathbf{m}. \quad (63)$$

We now choose a particular form of constitutive equation for the couple stress tensor. Specifically, we assume that it is fully antisymmetric and proportional to the electric current potential:

$$\boldsymbol{\mu} = \alpha \mathbf{H} \times \mathbf{I} \quad \Rightarrow \quad \nabla \cdot \boldsymbol{\mu} = \alpha \nabla \times \mathbf{H}, \quad (64)$$

where α just as in Sommerfeld's arguments is a factor of proportionality. Hence we obtain

$$-\frac{\partial \frac{\rho J}{\alpha} \boldsymbol{\omega}}{\partial t} + \nabla \times \mathbf{H} = -\frac{\rho}{\alpha} \mathbf{m}. \quad (65)$$

We now put:

$$\mathbf{D} = \frac{\rho J}{\alpha} \boldsymbol{\omega}, \quad \mathbf{j} = -\frac{\rho}{\alpha} \mathbf{m} \quad (66)$$

in order to arrive to a form of the Ørsted–Ampère law (50) specialized to the case of rest, $\mathbf{v} = \mathbf{0}$

$$-\frac{\partial \mathbf{D}}{\partial t} + \nabla \times \mathbf{H} = \mathbf{j}, \quad (67)$$

According to this the charge potential corresponds to the angular velocity and the current density to the body couple density. Note that in this mechanical model of the æther the electric current must not vanish. If we now take the divergence of the last equation and recall Equation (52) for the case of $\mathbf{v} = \mathbf{0}$ we arrive at Gauss' law:

$$-\frac{\partial}{\partial t}(\nabla \cdot \mathbf{D}) = -\mathbf{j} \cdot \frac{\partial \mathbf{q}}{\partial t} \quad \Rightarrow \quad \nabla \cdot \mathbf{D} = q. \quad (68)$$

It is therefore an indirect consequence of the balance of spin and the principle of conservation of charge, which goes beyond mechanics.

of matter, so one should not expect that they can be derived from the macroscopic properties of ponderable bodies. On the contrary, they seem to stem from the same root as general gravitation, that is, from the space-time metric, according to a recent paper by Schrödinger. ...Our remarks, however, may have some justification inasmuch as they show: if we were to construct an "ether" as a substratum for Maxwell's equation, then we would have to furnish it with qualities that are diametrically opposed to those of ordinary matter, viz., an *absolute directional orientation relative to space* in contrast to the relative orientation of the volume elements toward each other possessed by elastic bodies."

³⁸ Recall that Sommerfeld considered the velocity to be small, so that the second term in the material velocity was neglected. Here it is assumed that the æther is at rest in every inertial system.

We now investigate the balance of internal energy (30)₁ and subject it to our specializations, to which we add vanishing heat flux and radiation supply, $\mathbf{q} = \mathbf{0}$ and $r = 0$, respectively

$$\rho \frac{\partial u}{\partial t} = \boldsymbol{\mu} : \nabla \otimes \boldsymbol{\omega} \equiv -\frac{\alpha}{\mu_0} \mathbf{B} \cdot (\nabla \times \boldsymbol{\omega}). \quad (69)$$

The following quadratic form of stored ætheral elastic energy, so to speak, is proposed,³⁹

$$u = \frac{1}{2} \beta (\nabla \times \boldsymbol{\varphi})^2. \quad (70)$$

Because in our case $\boldsymbol{\omega} = \frac{d\boldsymbol{\varphi}}{dt} \equiv \frac{\partial \boldsymbol{\varphi}}{\partial t}$ it follows after insertion that

$$\mathbf{B} = -\frac{\rho \beta \mu_0}{\alpha} (\nabla \times \boldsymbol{\varphi}), \quad (71)$$

in words, the magnetic field corresponds mechanically to the curl of the twist angle, which has a certain intuitive appeal. Keeping this in mind one is automatically led to the law of non-existing magnetic monopoles, Equation (49). Note that this argument is similar to Sommerfeld's reasoning in context with Equation (57)₂. If we differentiate the last equation with respect to time and observe Equations (66)₁ and (53)₁ we obtain the Faraday law (48) with a constraint in the form:

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, \quad c = \sqrt{\frac{\beta}{J}}. \quad (72)$$

The latter is analogous to Sommerfeld's expression (61), the difference being that the microinertia of the æther takes over the role of its mass density.

In comparison of both methods:

- Within the approach based on micropolar theory Maxwell's equations can be “derived” based on the balances of spin and internal energy in combination with a constitutive equation for the couple stress tensor, which is linear in the current potential, and a quadratic expression w.r.t. the curl of the twist angle for the internal (stored) energy. The equation of linear momentum is not involved. However, the balance of charge is.
- Therefore, in contrast to Sommerfeld's approach, electric charges and electric currents are permitted.
- In Sommerfeld's approach use is made of the kinematic relation for the vorticity. The angular velocity does not appear. The micropolar approach is based on angular velocity instead. In the terminology of micropolar theory this means that in the first case the coupling coefficient (see [3]) is equal to one and in the second case equal to zero.

3 | CONCLUSIONS AND OUTLOOK

The objectives of this paper were manifold:

- Truesdell's classical papers on the moment of momentum were critically revisited and put into context with modern literature on generalized continua, in particular the modern theory of micropolar media.
- As fundamental laws (or axioms/postulates) of generalized continua the balances of mass, microinertia, linear momentum, total angular momentum, and total energy were established. Precepts such as the balance of moment of momentum or kinetic energy must be regarded as consequences of the fundamental law of linear momentum. Then, by subtrac-

³⁹ More generally we may write the following stored energy expression for a constrained medium, $\frac{1}{2} \boldsymbol{\varepsilon} : \mathbf{C} : \boldsymbol{\varepsilon} - \frac{k}{2} (\mathbf{I} \times \boldsymbol{\varphi}) : \boldsymbol{\varepsilon} + \frac{1}{2} (\mathbf{I} \times \boldsymbol{\varphi}) : \boldsymbol{\kappa} : (\mathbf{I} \times \boldsymbol{\varphi})$, where \mathbf{C} and $\boldsymbol{\kappa}$ are the stiffness tensors for translational deformation, $\boldsymbol{\varepsilon} = \frac{1}{2} (\nabla \otimes \mathbf{u} + \mathbf{u} \otimes \nabla)$, and for rotational deformation, $\mathbf{I} \times \boldsymbol{\varphi}$, respectively. We conclude that in his derivation Sommerfeld uses only the coupling energy term for his stress tensor in Equation (55). Whilst it is understandable that he neglects the pure rotational part in the energy, avoidance of the traditional translational stored energy should be attributed to a vanishing stiffness of the æther material, which is an unusual material property.

tion from the balances of total angular momentum and total energy, the balances of dynamic spin and internal energy will follow.

- Two examples of the power of the novel theories were given: (a) The possibilities resulting from a new field, \mathcal{B} , that couples linear and angular momentum; (b) Mechanical analogies of Maxwell's equations based on linear momentum or, alternatively, on the spin field.
- A critical examination of Newton's and Euler's statements in context with rigid bodies and moment of momentum.
- An analysis of the current state in mechanics and physics education regarding the use of moment of momentum, specifically for rigid bodies.

Several conclusions can be drawn, such as:

- A clear distinction between fundamental laws and precepts obtained from them must be made. Generally, such a precept can be obtained from a law only under some assumptions. It means that it is valid only in a certain situation (for example, for non-polar media, under the action of central forces, *etc.*). It is essential to be aware of its limitations.
- A deep understanding of the fundamentals of angular momentum is imperative, because otherwise internal contradictions during problem solving will inevitably arise.
- Every additional degree of kinematic freedom gives us additional possibilities to deal with more complicated (or more general) situations and processes.
- Mechanical analogies of physical phenomena and quantities can be suggested. However, all theories (mechanical, quantum, relativistic, *etc.*) are only models, and never cover the reality in full. Like creating a painting, sculpture, movie, or book they are a possible way to display and understand the real world rationally.

We conclude our excursion appropriately with a Latin proverb from the book of quotes by Anicius Manlius Torquatus Severinus Boëthius: "Si tacuisses, philopsophus mancisses!," which the readers are kindly asked to translate on their own.

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REFERENCES

- [1] Braun, M.: Balance of material momentum at a shock wave, ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik 89(8) 631–640 (2009)
- [2] Chandrasekhar, S.: Newton's Principia for the Common Reader. Clarendon Press, Oxford (1995)
- [3] Cowin, S.C.: The theory of polar fluids. In: Yih, C.S. (ed.) Advances in Applied Mechanics, vol. 14, pp. 279–347. Academic Press, Inc., New York (1974)
- [4] Dreyer, W., Guhlke, C., Müller, R.: Modeling of electrochemical double layers in thermodynamic non-equilibrium. Phys. Chem. Chem. Phys. 17(40), 27176–27194 (2015)
- [5] Eremeyev, V.A., Lebedev, L.P., Altenbach, H.: Foundations of Micropolar Mechanics. Springer Science & Business Media, Heidelberg (2012)
- [6] Eringen, A.C.: An assessment of director and micropolar theories of liquid crystals. Int. J. Eng. Sci. 31(4), 605–616 (1993)
- [7] Eringen, A.C.: Microcontinuum Field Theories: I Foundations and Solids. Springer Verlag, New York (1999)
- [8] Eringen, A.C.: Microcontinuum Field Theories: II Fluent Media. Springer Verlag, New York (2001)
- [9] Eringen, A.C.: Nonlocal Continuum Field Theories. Springer-Verlag, New York (2002)
- [10] Eringen, A.C., Kafadar, C.B.: Polar field theories. In: Continuum Physics IV. Academic Press, London (1976)
- [11] Eringen, C.: Mechanics of Continua. Robert E. Krieger Publishing Company, New York (1980)
- [12] Euler, L.: Nova methodus motum corporum rigidorum determinandi. Novi Comm. Acad. Sci. Petrop. 20, 208–238 (1776)
- [13] Feynman, R.P., Leighton, R.B., Sands, M.: The Feynman Lectures on Physics, vol. 1, Mainly Mechanics, Radiation, and Heat. Addison-Wesley Publishing Company, Reading (1963)
- [14] Fosdick, R., Royer-Carfagni, G.: Stress as a constraint reaction in rigid bodies. J. Elast. 74(3), 265–276 (2004)
- [15] Frolova, K., Vilchevskaya, E., Polyanskiy, V., Alekseeva, E.: Modelling of a hydrogen saturated layer within the micropolar approach. In: Abali, B., Altenbach, H., dell'Isola, F., Eremeyev, V., Öchsner, A. E. (eds.) New Achievements in Continuum Mechanics and Thermodynamics, pp. 117–128. Springer, Cham (2019)
- [16] Goldstein, H., Poole, C., Safko, J.: Classical Mechanics, 3rd edn. Addison-Wesley Publishing Company, San Francisco (2002)

- [17] Grioli, G.: On the stress in rigid bodies. *Meccanica* 18(1), 3–7 (1983)
- [18] Gross, D., Hauger, W., Schröder, J., Wall, W.A., Govindjee, S.: *Engineering Mechanics 3: Dynamics*. Springer, Heidelberg (2011)
- [19] Hamel, G.: *Theoretische Mechanik, Eine einheitliche Einführung in die gesamte Mechanik, Berichtigter Nachdruck*. Springer, Heidelberg (1967)
- [20] Heaviside, O.: *Electrical Papers in two Volumes, vol. 1*. Macmillan and Co., London (1892)
- [21] Herivel, J.W.: *The Background to Newton's Principia*. Clarendon Press, Oxford (1965)
- [22] Hibbeler, R.: *Engineering Mechanics Dynamics*, 12th edn. Prentice Hall, Upper Saddle River (2010)
- [23] Hutter, K., Jöhnk, K.: *Continuum Methods of Physical Modeling. Continuum Mechanics, Dimensional Analysis, Turbulence*. Springer, Heidelberg (2004)
- [24] Hutter, K., Ven, A.A., Ursescu, A.: *Electromagnetic Field Matter Interactions in Thermoelastic Solids and Viscous Fluids, vol. 710, Lecture Notes in Physics*. Springer, Heidelberg (2007)
- [25] Hutter, K., Wang, Y.: *Fluid and Thermodynamics, Volume III: Structured and Multiphase Fluids*. Springer, Cham (2018)
- [26] Ivanova, E.A.: A new model of a micropolar continuum and some electromagnetic analogies. *Acta Mech.* 226(3), 697–721 (2015)
- [27] Ivanova, E.A.: On a micropolar continuum approach to some problems of thermo- and electrodynamics. *Acta Mech.* 230(5), 1685–1715 (2019)
- [28] Ivanova, E.A.: Towards micropolar continuum theory describing some problems of thermo- and electrodynamics. In: Altenbach, H., Irschik, H., Matvienko, V. (eds.) *Contributions to Advanced Dynamics and Continuum Mechanics*, pp. 111–129. Springer, Cham (2019)
- [29] Ivanova, E.A., Kolpakov, Y.E.: The use of moment theory to describe the piezoelectric effect in polar and non-polar materials. In: Altenbach, H., Forest, S. (eds.) *Generalized Continua as Models for Materials*, pp. 163–178. Springer, Heidelberg (2013)
- [30] Ivanova, E.A., Matias, D.V.: Coupled problems in thermodynamics. In: Altenbach, H., Oechsner, A. (eds.) *State of the Art and Future Trends in Material Modeling*, pp. 151–172. Springer, Cham (2019)
- [31] Ivanova, E.A., Vilchevskaya, E.N.: Micropolar continuum in spatial description. *Contin. Mech. Thermodyn.* 28(6), 1759–1780 (2016)
- [32] Ivanova, E.A., Vilchevskaya, E.N., Müller, W.H.: Time derivatives in material and spatial description — What are the differences and why do they concern us? In: Naumenko, K., Aßmus, M. (eds.) *Advanced Methods of Mechanics for Materials and Structures, vol. 60*, pp. 3–28. Springer, Singapore (2016)
- [33] Jammer, M.: *Concepts of Force*. Harvard University Press, Massachusetts (1957)
- [34] Joos, G.: *Theoretical Physics*. Stechert, New York (1934)
- [35] Kittel, C., Knight, W.D., Ruderman, M.A.: *Berkeley Physics Course, Mechanics, vol. 1, 2nd edn*. McGraw-Hill, Inc., New York (1973)
- [36] Kovetz, A.: *Electromagnetic Theory*. Oxford University Press, Oxford (2000)
- [37] Koyré, A., Cohen, I.B., Whitman, A.: *Isaac Newton's Philosophiæ Naturalis Principia Mathematica, 3rd edn (1726) with variant readings, vol. I/II*. Cambridge University Press, Cambridge (1972)
- [38] Landau, L.D., Lifshitz, E.M.: *Mechanics 3rd edn, Course of Theoretical Physics, vol. 1*. Butterworth-Heinemann, Oxford (2000)
- [39] Leibfried, G.: *Handbuch der Physik, vol. VII/1, Kristallphysik, Gittertheorie der mechanischen und thermischen Eigenschaften der Kristalle*, pp. 104–324. Springer, Berlin (1955)
- [40] Lindner, A., Strauch, D.: *A Complete Course on Theoretical Physics, Undergraduate Lecture Notes in Physics*. Springer Nature, Switzerland AG (2018)
- [41] Maugin, G.A.: Generalized continuum mechanics: What do we mean by that? In: Maugin, G.A., Metrikine, A.V. (eds.) *Mechanics of Generalized Continua One Hundred Years after the Cosserats*, pp. 3–13. Springer, Berlin (2010)
- [42] Maugin, G.A.: A historical perspective of generalized continuum mechanics. In: Altenbach, H., Maugin, G.A., Erofeev, V. (eds.) *Mechanics of Generalized Continua*, pp. 3–16. Springer, Berlin (2011)
- [43] Maxwell, J.C.: LI. On physical lines of force. Part II.—The theory of molecular vortices applied to electric currents. *London, Edinburgh, Dublin Philos. Mag. J. Sci.* 21(141), 338–348 (1861)
- [44] Maxwell, J.C.: XXV. On physical lines of force. Part I.—The theory of molecular vortices applied to magnetic phenomena. *London, Edinburgh, Dublin Philos. Mag. J. Sci.* 21(139), 161–175 (1861)
- [45] Maxwell, J.C.: III. On physical lines of Force. Part III.—The theory of molecular vortices applied to statical electricity. *London, Edinburgh, Dublin Philos. Mag. J. Sci.* 23(151), 12–24 (1862)
- [46] Maxwell, J.C.: XIV. On physical lines of force. Part IV.—The theory of molecular vortices applied to the action of magnetism on polarized light. *London, Edinburgh, Dublin Philos. Mag. J. Sci.* 23(141), 85–95 (1862)
- [47] Maxwell, J.C.: VIII. A dynamical theory of the electromagnetic field. *Philos. Trans. R. Soc. Lond. A, Math. Phys. Sci.* 155, 459–512 (1865)
- [48] Maxwell, J.C.: *Ether*. In: *Encyclopædia Britannica, 9th edn*, pp. 568–572. Charles Scribner's Sons, New York (1878)
- [49] Müller, I.: *Thermodynamics, Interaction of Mechanics and Mathematics Series*. Pitman, London (1985)
- [50] Müller, I., Müller, W.H.: *Fundamentals of Thermodynamics and Applications: With Historical Annotations and Many Citations from Avogadro to Zermelo*. Springer Science & Business Media, New York (2009)
- [51] Müller, I., Ruggeri, T.: *Rational Extended Thermodynamics, vol. 37, 2nd edn*. Springer Science & Business Media, New York (1998)
- [52] Müller, W.H.: *An Expedition to Continuum Theory, Solid Mechanics and its Applications Series*. Springer, Berlin (2014)
- [53] Müller, W.H., Vilchevskaya, E.N.: Micropolar theory from the viewpoint of mesoscopic and mixture theories. *Phys. Mesomech.* 20(3), 263–279 (2017)

- [54] Müller, W.H., Vilchevskaya, E.N.: Micropolar theory with production of rotational inertia: A rational mechanics approach. In: Altenbach, H., Pouget, J., Rousseau, M., Collet, B., Michelitsch, T. (eds.) *Generalized Models and Non-classical Approaches in Complex Materials 1*, pp. 195–229. Springer, Cham (2018)
- [55] Müller, W.H., Vilchevskaya, E.N., Weiss, W.: A meso-mechanics approach to micropolar theory: A farewell to material description. *Phys. Mesomech.* 20(3), 250–262 (2017)
- [56] Newton, I., Motte, A., Chittenden, N.W.: *Newton's Principia: The Mathematical Principles of Natural Philosophy*. Daniel Adee, New York (1846)
- [57] Pap, A.: *The A Priori in Physical Theory*. Springer, New York (1946)
- [58] Pourciau, B.: Newton's interpretation of newton's second law. *Arch. Hist. Exact Sci.* 60(2), 157–207 (2006)
- [59] Rivlin, R.: Generalized mechanics of continuous media. In: Kröner, E. (ed.) *Mechanics of Generalized Continua*, Proceedings of IUTAM Symposium Freudenstadt-Stuttgart, pp. 1–17. Springer-Verlag, Heidelberg (1968)
- [60] Sedov, L.: *A Course in Continuum Mechanics*, vol. I, Basic Equations and Analytical Techniques. Wolters-Noordhoff Publishing Company, Groningen (1971)
- [61] Simon, H.A.: Xciii. the axioms of newtonian mechanics. *London, Edinburgh, Dublin Philos. Mag. J. Sci.* 38(287), 888–905 (1947)
- [62] Sommerfeld, A.: *Lectures on Theoretical Physics*, vol. II, *Mechanics of Deformable Bodies*. Academic Press Inc., New York (1950)
- [63] Sommerfeld, A.: *Vorlesungen über theoretische Physik*, 5. Auflage. Akademische Verlagsgesellschaft, Leipzig (1955)
- [64] Sommerfeld, A.: *Vorlesungen über theoretische Physik*, Band II, *Mechanik der deformierbaren Medien*. Verlag Harri Deutsch, Thun, Frankfurt/M (1992)
- [65] Steigmann, D.J.: On the formulation of balance laws for electromagnetic continuask. *Math. Mech. Solids* 14(4), 390–402 (2009)
- [66] Stewart, I.W.: *The Static and Dynamic Continuum Theory of Liquid Crystals: A Mathematical Introduction*. Taylor & Francis, London (2004)
- [67] Szabó, I.: *Geschichte der mechanischen Prinzipien und ihrer wichtigsten Anwendungen*. Birkhäuser, Cham (1977)
- [68] Thomson, W., Tait, P.: *Treatise on natural philosophy*, Part II. Cambridge University Press, Cambridge (1912)
- [69] Truesdell, C.: Zusammenfassender Bericht Die Entwicklung des Drallsatzes. *Z. Angew. Math. Mech.* 44(4–5), 149–158 (1964)
- [70] Truesdell, C.: *Essays in the History of Mechanics*. Springer, Heidelberg (1968)
- [71] Truesdell, C., Toupin, R.: *Handbook of Physics: The Classical Field Theories*, pp. 226–858. Springer-Verlag, Heidelberg (1960)
- [72] Vilchevskaya, E.: Micropolar theory with inertia production. In: Altenbach, H., Öchsner, A. (eds.) *Advanced Structured Materials*, vol. 100, pp. 421–442. Springer Nature, Cham (2019)
- [73] Voigt, W.: Theoretische studien über die elasticitätsverhältnisse der kristalle. *Göttinger Abhandlungen* 34, 3–100 (1887)
- [74] Zhilin, P.A.: Теоретическая механика (Фундаментальные законы механики) (Theoretical mechanics (Fundamental laws of mechanics)), Санкт-Петербург Издательство Политехнического университета, St. Petersburg (2001)
- [75] Zhilin, P.A.: Рациональная механика сплошных сред (Rational Continuum Mechanics) Санкт-Петербург Издательство Политехнического университета, St. Petersburg (2012)
- [76] Zhilin, P.A.: Построение модели электромагнитного поля с позиций рациональной механики (Construction of a model of an electromagnetic field from the standpoint of rational mechanics), РЭНСИТ 5(1), 77–97 (2013)
- [77] Zhilin, P.A.: Динамика твердого тела (Dynamics of the Solid Body) Санкт-Петербург Издательство Политехнического университета, St. Petersburg, (2015).

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APPENDIX A: DERIVATION OF EQUATIONS (44)–(47)

Under the assumptions made in Section 2.4 the balance of linear momentum and the spin balance reduce to

$$m \frac{d}{dt}(\mathbf{v} + B\boldsymbol{\omega}) = \mathbf{0}, \quad m \frac{d}{dt}(J\boldsymbol{\omega} + B\mathbf{v}) = -mB\mathbf{v} \times \boldsymbol{\omega}. \quad (\text{A.1})$$

The first equation is easily solved by $\mathbf{v} = B(\boldsymbol{\omega}_0 - \boldsymbol{\omega}) + \mathbf{v}_0$. Therefore, the second equation can be uncoupled,

$$\frac{d\boldsymbol{\omega}}{dt} = \frac{\alpha}{a} \mathbf{a} \times \boldsymbol{\omega}, \quad \mathbf{a} = \mathbf{v}_0 + B\boldsymbol{\omega}_0, \quad a = \|\mathbf{a}\|, \quad \alpha = \frac{Ba}{B^2 - J}. \quad (\text{A.2})$$

Equation (A.2) represents a linear system of ordinary differential equations. Its solution is found by making the *ansatz* $\boldsymbol{\omega} = \exp(\lambda t)\mathbf{r}$. Note that although this expression has a constant direction for $\boldsymbol{\omega}$ and a changing amplitude, the solution to the system of differential equations may change its direction in time in the end. By means of the *ansatz* the problem is

converted into an eigenvalue problem

$$\mathbf{A} \cdot \mathbf{r} = \lambda \mathbf{r}, \quad \mathbf{A} = -\frac{\alpha}{a} \mathbf{a} \cdot \boldsymbol{\epsilon}^{(3)}, \quad (\text{A.3})$$

where $\boldsymbol{\epsilon}^{(3)}$ is the Levi-Civita tensor. By introducing the orthonormal basis,

$$\mathbf{b}_1 = \frac{\mathbf{a}}{a}, \quad \mathbf{b}_2 = \frac{\mathbf{a} \times \boldsymbol{\omega}_0}{\|\mathbf{a} \times \boldsymbol{\omega}_0\|}, \quad \mathbf{b}_3 = \frac{\mathbf{a} \times (\mathbf{a} \times \boldsymbol{\omega}_0)}{\|\mathbf{a} \times (\mathbf{a} \times \boldsymbol{\omega}_0)\|}, \quad a = \|\mathbf{a}\| \quad (\text{A.4})$$

the tensor eigenvalue problem is reduced to a matrix eigenvalue problem:

$$\mathbf{A} = \alpha \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}_{ij} \quad \mathbf{b}_i \otimes \mathbf{b}_j = B_{ij} \mathbf{b}_i \otimes \mathbf{b}_j, \quad \mathbf{r} = r_i \mathbf{b}_i \Rightarrow B_{ij} r_j = \lambda r_i. \quad (\text{A.5})$$

With the standard methods available from linear algebra one obtains the solution:

$$\lambda_1 = 0, \quad \mathbf{r}_1 = \mathbf{a}, \quad \lambda_{2/3} = \pm i\alpha, \quad \mathbf{r}_{2/3} = \pm i\mathbf{a} \times \boldsymbol{\omega}_0 + a^{-1} \mathbf{a} \times (\mathbf{a} \times \boldsymbol{\omega}_0), \quad (\text{A.6})$$

where the eigenvectors were scaled conveniently. Since there are three linearly independent solutions, the general solution to the system of ordinary differential equations is given by their linear combination

$$\boldsymbol{\omega} = c_1 \mathbf{a} + c_2 \exp(i\alpha t) \mathbf{a} \times [\mathbf{a} \times \boldsymbol{\omega}_0 + i\mathbf{a} \boldsymbol{\omega}_0] + c_3 \exp(-i\alpha t) \mathbf{a} \times [\mathbf{a} \times \boldsymbol{\omega}_0 - i\mathbf{a} \boldsymbol{\omega}_0]. \quad (\text{A.7})$$

By observing the initial condition $\boldsymbol{\omega}(t=0) = \boldsymbol{\omega}_0$ one finds $c_1 = a^{-2} \mathbf{a} \cdot \boldsymbol{\omega}_0$ and $c_2 = c_3 = -\frac{1}{2a^2}$. By using Euler's formula for complex exponentials we find

$$\begin{aligned} \boldsymbol{\omega}(t) &= a^{-2} (\mathbf{a} \cdot \boldsymbol{\omega}_0) \mathbf{a} - a^{-2} [\cos(\alpha t) \mathbf{a} \times (\mathbf{a} \times \boldsymbol{\omega}_0) - \sin(\alpha t) \mathbf{a} \mathbf{a} \times \boldsymbol{\omega}_0] \\ &= [\cos(\alpha t) \mathbf{I} + (1 - \cos(\alpha t)) \mathbf{n} \otimes \mathbf{n} + \sin(\alpha t) \mathbf{n} \times \mathbf{I}] \cdot \boldsymbol{\omega}_0, \quad \mathbf{n} = \frac{\mathbf{a}}{a}. \end{aligned} \quad (\text{A.8})$$

and for the velocity

$$\mathbf{v}(t) = \mathbf{v}_0 - B[(\cos(\alpha t) - 1) \mathbf{I} + (1 - \cos(\alpha t)) \mathbf{n} \otimes \mathbf{n} + \sin(\alpha t) \mathbf{n} \times \mathbf{I}] \cdot \boldsymbol{\omega}_0, \quad \mathbf{n} = \frac{\mathbf{a}}{a}. \quad (\text{A.9})$$

The integration of this expression w.r.t. time is simple and leads to the third Equation (47).